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ON A MULTI-DIMENSIONAL SINGULAR STOCHASTIC CONTROL PROBLEM: THE PARABOLIC CASE

by

NGUYEN DO MINH NHAT

DISSERTATION

Submitted to the Graduate School

of Wayne State University,

Detroit, Michigan

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

2015

MAJOR: MATHEMATICS

Approved By:

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Advisor Date

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DEDICATION

To my family

To my teachers

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First and foremost I am grateful to my advisor, Professor Jose Luis Menaldi. Throughout the years of my graduate study at WSU, I have been benefited enormously from his ideas, enthusiasm and overall guidance. I thank him for not only serving as my research supervisor and spending countless hours in sharing his broad knowledge of mathematics with me, but also for his encouragement, patience and consideration during the years.

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Finally, I thank my wife Vy and my daughters Selena and Brianna for their love, understanding and tolerance. To them this thesis is dedicated.

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GENERAL NOTATION

$|A|$: Lebesgue measure of A.

Ω : a bounded domain in $\mathbb{R}^n$.

Γ : $\partial\Omega$.

Σ : $\Gamma\times]0,T[,\ 0<T<\infty$.

$\mathcal{D}(\Omega)$: the set of indefinitely differentiable functions with compact support in Ω.

Q : $\Omega\times(0,T)$.

$\partial_p Q$: $\partial\Omega\times(0,T)\times\bar{\Omega}\times\{0\}$.

$W^{k,p}(Q)$: denote the Sobolev space of real valued functions on Q

whose generalized derivatives of order less than or equal to k are in $L^p(Q), 1\le p\le\infty$.

$H^k(Q)$: $W^{k,2}(Q)$ for $k=0,1,2,\cdots$

$H_0^1(Q)$: denote the closure of $C_0^\infty(Q)$ in $H^1(Q)$.

$(\cdot,\cdot)$: denote the inner product in $L^2(Q)$.

$L^p([a,b];\mathbb{R}^n)$: the family of Borel measurable function $h:[a,b]\to\mathbb{R}^n$ such that $\int_a^b |h(t)|^p dt<\infty$.

$\mathcal{L}^p([a,b];\mathbb{R}^n)$: the family of $\mathbb{R}^n$ -value $\mathcal{F}_t$ -adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b |f(t)|^p dt<\infty$ a.s.

$\mathcal{M}^p([a,b];\mathbb{R}^n)$: the family processes $\{f(t)\}_{a\le t\le b}$ in $\mathcal{L}^p([a,b];R^n)$ such that $\mathbb{E}\int_a^b |f(t)|^p dt<\infty$.

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CHAPTER 1 INTRODUCTION

Stochastic control is the study of dynamical systems subject to random perturbations and which can be controlled in order to optimize some performance criteria. Over the past six decades, stochastic control has been developed extensively with many applications to other sciences, engineering, finance, economics, etc. In the stochastic control theory, there are many very interesting problems and a variety of approaches has been proposed to show these problems. Let us list here some of them among others.

1.1 Singular stochastic control

The class of singular stochastic control problems, which has been studied extensively in recent years, deals with systems described by a stochastic differential equation in which one restricts the cumulative displacement of the state caused by control to be of an additive nature. More precisely, in singular control problems the state process is governed by the following n-dimensional stochastic differential equation

$$x_s = x + \int_t^s b(\theta, x_\theta, u_\theta)d\theta + \int_t^s \sigma(\theta, x_\theta, u_\theta)dB_\theta + \int_t^s g(\theta)dv_\theta \tag{1.1}$$

on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $b(\cdot,\cdot,\cdot), \sigma(\cdot,\cdot,\cdot), g(\cdot)$ are given deterministic functions, $(B_s, s \geq 0)$ is a n-dimensional Brownian motion, x is the initial state at time t, and $u : [0,T] \to U, v : [0,T] \to \mathbb{R}^k$ with v nondecreasing componentwise, stand for the controls, U is called the control set. The expected cost has the form

$$J(u,v) = \mathbb{E}\Big\{ \int_t^T f(s, x_s, u_s)ds + \int_{[t,T)} c(s)dv_s \Big\}, \tag{1.2}$$

where $f(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}, c(\cdot) : [0,T] \to \mathbb{R}^k_+$ are given, and f stands for the running cost rate of the problem and c the cost rate of applying the singular control. Some special cases of the one dimensional problem of this type have been studied by many authors including Bather

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and Chernoff [7], [8], Beněs *et al.* [10], Borodowski *et al.* [18], Bratus [21], Karatzas [52], [51], Chow *et al.* [26], El Karoui and Karatzas [50], Harrison and Taksar [43], Karatzas and Shreve [53], Lehoczky and Shreve [60], Ma [66], Menaldi and Robin [71], [72], and Sun [83]. It is shown that the value function satisfied a variational inequality which gives rise to a free boundary problem, and the optimal state process is a diffusion reflected at the free boundary. This approach encounters substantial difficulties for the problems in high dimension due to the lack of information about the regularity of the associated free boundary. In Soner and Shreve [82], a special two dimensional problem ($b = 0, \sigma = I$) was considered. It was shown there that the associated free boundary is smooth enough to construct a reflected diffusion in the continuation region. However, the method depends heavily on the special features of the problem and cannot be extended to general problems. Another result about high dimensional problems can be found in Menaldi and Taksar [73], who considered the n-dimensional case with $b =$ const, $\sigma =$ const. It was shown that the value function satisfies the associated Hamilton-Jacobi-Bellman (HJB) equation, and the existence of the optimal control was proved without requiring any regularity about the free boundary. By applying the compactification method used in Haussmann [45], Haussmann and Lapeltier [46], and El Karoui *et al.* [54], then the existence of the optimal control can be shown, see Haussmann and Suo [47]. In William *et al.* [89], the regularity question has been solved partially, with certain assumptions the free boundary is smooth away from some 'corner points'. By using weak convergence arguments and a time rescaling technique, the existence of an optimal control with state constraints has been established in Amarjit and Kevin [23]. Besides, a class of singular stochastic control problem with recursive utility where the cost function is determined by a backward stochastic differential equation (BSDE) has been studied in Wang [88]. The stochastic control problems with recursive utility was firstly introduced by Peng [77], in which the author studied an absolutely continuous stochastic control problem with recursive utility where the cost function is determined by a BSDE. In recent years, it has been widely recognized that this class of control problems provides a useful framework in mathematical finance and differential games (see for example Duffie and Epstein [31], El Karoui

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et al. [32], Hamadéne and Lepeltier [41]).

1.2 Other control problems

Moreover, there are some other control problems that are also significant theoretical and practical interest. We will list some of those and also emphasize some present developments.

1.2.1 Random horizon

In problem formulation (1.2), the time horizon is fixed until a deterministic terminal time T. In some real applications, the time horizon may be random, the control problem is formulated as:

$$\sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big\{ \int_0^\tau f(s, X_s, \alpha_s)ds + g(X_\tau)\Big\}, \tag{1.3}$$

where τ is a finite random time, f and g are given functions defined on $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ and $\mathbb{R}^n$. In standard cases, the terminal time τ is a stopping time at which the state process exits from a certain relevant domain. For example, in a reinsurance model, the state process X is the reserve of a company that may control it by reinsuring a proportion $1-\alpha$ of premiums to another company. The terminal time τ is then the bankruptcy time of the company defined as $\tau = \inf\{t \geq 0 : X_t \leq 0\}$. More generally, given some open set $\mathcal{O}$ of $\mathbb{R}^n$,

$$\tau = \inf\{t \geq 0 : X_t \notin \mathcal{O}\} \wedge T$$

(which depends on the control). In this case, the control problem (1.3) leads via the dynamic programming approach to a Dirichlet boundary-value problem. Another case of interest concerns a terminal time τ , which is a random time but not a stopping time in the filtration F with respect to which the controls are adapted. This situation occurs, for example, in credit risk models where t is the default time of a firm. Under the so-called hypothesis on filtration theory, $P[\tau \leq t|\mathcal{F}_t]$ is a nondecreasing right-continuous process, problem (1.3) may be reduced to a stochastic control problem under a fixed deterministic horizon, see Blanchet-Scalliet *et al.* [17] for a recent application in portfolio optimization model. In the general random time case, the associated control problem has

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been studied in the literature, see Bouchard and Pham [19] or Zitkovic [92] for a utility maximization problem in finance.

1.2.2 Optimal Stopping

In the models presented above, the horizon of the problem is either fixed or indirectly influenced by the control. When one has the possibility to control directly the terminal time, which is then modeled by a controlled stopping time, the associated problem is an optimal stopping time problem. In the general formulation of such models, the control is mixed, composed by a pair control/stopping time (α, τ) and the functional to optimize is:

$$\mathbb{E}\Big\{ \int_0^\tau f(t, X_t, \alpha_t)dt + g(X_\tau) \Big\}.$$

The theory of optimal stopping has received an interest with a variety of applications in economics and finance. These applications range from asset pricing (American options) to firm investment and real options. Extensions of classical optimal stopping problems deal with multiple optimal stopping with eventual changes of regimes in the state process. They were studied, e.g. in Bensoussan and Lions [13], Tang and Yong [84], and applied in finance in Brekke and Oksendal [22], Duckworth and Zervos [30], Guo [40].

1.2.3 Impulse Control

In formulation of the control problem, the displacement of the state changes continuously in time in response to the control effort. However, in many real applications, this displacement may be discontinuous. For example, in insurance company models, the company distributes the dividends once or twice a year rather than continuously. In transaction costs models, the agent should not invest continuously in the stock due to the costs but only at discrete times. A similar situation occurs in a liquidity risk model, see e.g. Cetin *et al.* [25]. Impulse control provides a suitable framework for modeling such situations. This may be described as follows: the controlled state diffusion process

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is governed by

$$dX_s = b(s, X_s)dt + \sigma(s, X_s)dW_s + d\zeta_s,$$

where the control ζ is a pure jump process. In other words, the control is given by a pair $(\tau_n, \kappa_n)_n$ where $(\tau_n)_n$ is a nondecreasing sequence of stopping times representing the intervention times of the controller, and $(\kappa_n)_n$ is a sequence of $\mathcal{F}_{\tau_n}$-measurable random variables representing the jump size decided by the controller at time (τ_n). The functional objective to optimize is in the form:

$$\mathbb{E}\Big\{\int_0^T f(t, X_t, \alpha_t)dt + \sum_{\tau_n \le T} h(X_{\tau_n}, \kappa_n) + g(X_T)\Big\}.$$

Impulse control problem is known to be associated via the dynamic programming approach to an HJB quasi-variational inequality, see Bensoussan and Lions [13]. For some recent applications in finance, we refer to Jeanblanc and Shiryaev [49] for insurance models, Korn [56] and Oksendal and Sulem [75] for transaction costs models, and Ly Vath *et al.* [65] for liquidity risk model.

1.2.4 Optimal switching

Here one is allowed to switch the control $u(\cdot)$ at stopping times $\{\tau_i\}$ from $u(\tau_i-)$(the value immediately before τ_i) to a new (non anticipative) value $u(\tau_i)$ resp., with an associated cost $q(u(\tau_i), u(\tau_i-))$. The aim is to minimize

$$\mathbb{E}\Big\{\int_0^T e^{-\int_0^t c(X(s),u(s))ds} f(X_t, u_s)dt + \sum_{\tau_i \le T} e^{-\int_0^{\tau_i} c(X(s),u(s))ds} q(u(\tau_i), u(\tau_i-)) + e^{-\int_0^T c(X(s),u(s)ds} h(X(T))\Big\},$$

over reset times $\{\tau_i\}$, and reset values $\{u(\tau_i)\}$. Assume $q \ge \delta$ for some $\delta > 0$ to avoid infinitely many switchings in a finite time interval. The switching control can be considered in Menaldi *et al.* [69].

1.2.5 Ergodic control

Some stochastic systems may exhibit over a long period a stationary behavior characterized by an invariant measure. This measure, if it does exist, is obtained by the average of the states over

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a long time. An ergodic control problem consists in optimizing over the long term some criterion taking into account this invariant measure. A standard formulation is to optimize over control a functional of the form

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big\{\int_0^T f(X_t,\alpha_t)dt\Big\},$$

or

$$\limsup_{T\to\infty} \frac{1}{T}\ln \mathbb{E}\Big\{\exp\int_0^T f(X_t,\alpha_t)dt\Big\}.$$

This last formulation is called a risk-sensitive control on an infinite horizon. The singular control for multidimensional Gaussian-Poisson processes with a long-run (or ergodic) and a discounted criteria is discussed in Menaldi [70]. Ergodic and risk-sensitive control problems were studied in Karatzas [53], Bensoussan and Nagai [14] or Fleming and Rishel [34]. Risk sensitive control problems have been recently applied in a financial context in Bielecki and Pliska [16] and Fleming and Sheu [35]. Another criterion is based on the large deviations behavior of the ergodic system: $P[X_T/T] \simeq e^{-I(c)T}$, when T goes to infinity, consists in maximizing over control a functional of the form:

$$\limsup_{T\to\infty} \frac{1}{T}\ln P\Big[\frac{X_T}{T}\geq c\Big].$$

This large deviations control problem is interpreted in finance as the asymptotic version of the quantile criterion of maximizing the probability that the terminal wealth X_T beats a given benchmark. This nonstandard control problem has been introduced and developed recently by Pham [78], [79]. It does not have a direct dynamic programming principle but may be reduced via a duality principle to a risk-sensitive control problem.

1.2.6 Partial observation control problem

It is assumed so far that the controller completely observes the state system. In many real applications, one is only able to observe partially the state via other variables and there is noise in the observation system. For example in financial models, one may observe the asset price but

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not completely its rate of return and/or its volatility, and the portfolio investment is based only on the asset price information. We are facing a partial observation control problem. This may be formulated in a general form as follows : we have a controlled signal (unobserved) process governed by

$$dX_s = b(s, X_s, Y_s, \alpha_s)ds + \sigma(s, X_s, Y_s, \alpha_s)dW_s,$$

and an observation process

$$dY_s = \eta(s, X_s, Y_s, \alpha_s)ds + \gamma(s, X_s, Y_s, \alpha_s)dB_s,$$

where B is another Brownian motion, eventually correlated with W. The control α is adapted with respect to the filtration generated by the observation $\mathbb{F}^Y = (\mathcal{F}_t^Y)$ and the functional to optimize is:

$$J(\alpha) = \mathbb{E}\Big\{ \int_0^T f(X_t, Y_t, \alpha_t)dt + g(X_T, Y_T)\Big\}.$$

By introducing the filter measure-valued process

$$\Pi_t(dx) = P[X_t \in dx | \mathcal{F}_t^Y],$$

one may rewrite the functional $J(\alpha)$ in the form:

$$J(\alpha) = \mathbb{E}\Big\{ \int_0^T \hat{f}(\Pi_t, Y_t, \alpha_t)dt + \hat{g}(\Pi_T, Y_T)\Big\},$$

where we use the notation: $\hat{f}(\pi, y) = \int f(x, y)\pi(dx)$ for any finite measure π on the signal state space, and similarly for $\hat{g}$. Since by definition, the process (Π_t) is $(\mathcal{F}_t^Y)$ -adapted, the original partial observation control problem is reformulated as a complete observation control model, with the new observable state variable defined by the filter process. The additional main difficulty is that the filter process is valued in the infinite-dimensional space of probability measures : it satisfies the Zakai stochastic partial differential equation. The dynamic programming principle or maximum principle are still applicable and the associated Bellman equation or Hamiltonian system are now in infinite dimension. For a theoretical study of optimal control under partial observation under

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this infinite dimensional viewpoint, we mention among others the works Fleming [33], Davis and Varaiya [28], Baras *et al.* [6], Bensoussan [11], Lions [62] or Zhou [90]. There are relatively few explicit calculations in the applications to finance of partial observation control models and this area should be developed in the future.

1.3 Examples

Here we sketch in brief some recent applications of stochastic control.

1.3.1 Forest harvesting problem:

In this problem Alvarez [3], the so called 'stochastic forest stand value growth' is described up to extinction time γ by

$$X(t) = x + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s) - \sum_{\tau_k \leq \tau} \zeta_k,$$

where $\gamma = \inf\{t \geq 0 : X(t) \leq 0\}$ (possibly ∞) and the non-negative, non-anticipative random variables $\{\tau_k\}, \{\zeta_k\}$ are respectively the cutting times and the quantities cut at the respective cutting times. The aim is to maximize the forest revenue $\mathbb{E}\Big\{ \sum_{\tau_k \leq \gamma} e^{-r\tau_k}(X(\tau_k) - c)\Big\}$, where $c > 0$ is the reforestation cost and $r > 0$ is the discount factor. This is an impulse control problem.

1.3.2 Portfolio optimization:

In Korn and Kraft [56], the wealth process in portfolio optimization satisfies the s.d.e

$$dX(t) = X(t)[\pi(t)\mu(t) + (1 - \pi(t))r(t)dt + \pi(t)\sigma(t)dW(t)],$$

where $\mu(\cdot), \sigma(\cdot)$ are known and $\pi(\cdot)$ is the $[0, 1]$–value control process that specifies the fraction invested in the risky asset, the remaining wealth being invested in a bond. Here $r(\cdot)$ is a fluctuating interest rate process satisfying

$$dr(t) = a(t)dt + bdW'(t).$$

Both $a(\cdot)$ and b are assumed to be known and $W'(t)$ is a Brownian motion independent of $W(\cdot)$. The aim is to maximize $\mathbb{E}\{X(T)^\gamma\}$ for some $T, \gamma > 0$. An alternative 'mean variance' formulation in

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the spirit of Markowitz seeks to maximize a linear combination of the mean and negative variance of $X(T)$ Zhou and Li [91]. A 'risk sensitive' version of the problem, on the other hand, seeks to maximize

$$\liminf_{T\uparrow\infty} -\frac{2}{\theta T}\log \mathbb{E}\{e^{-(2/\theta)X(T)}\}.$$

See [56], Kuroda and Nagai [57] for more general formulation.

1.3.3 Production planning:

In Bensoussan *et al.* [15], considering a factory producing a single good. Let $y(\cdot)$ denote its inventory level as a function of time, $p(\cdot) \geq 0$ the production rate, ξ denote the constant demand rate and y_1, p_1 denote the factory-optimal inventory level and production rate respectively. The inventory process is modeled as the controlled diffusion

$$dy(t) = (p(t) - \xi)dt + \sigma dW(t),$$

where σ is a constant. The aim is to minimize over non-anticipative $p(\cdot)$ the discounted cost

$$\mathbb{E}\Big\{\int_0^\infty e^{-\alpha t}[c(p(t)-p_1)^2 + h(y(t)-y_1)^2]dt\Big\}$$

where c, h are known coefficients for the production cost and the inventory holding cost, resp.

1.3.4 Heavy traffic limits of queues:

The following control problem in Harrison and Zeevi [44] arises in the so called Halfin-Whitt limit of multi-type multi-server queues: Consider a system of d customer classes being jointly served by N identical servers, with $\lambda_i, \mu_i, \gamma_i$ denoting the respective arrival, service and per customer abandonment rates for class i. Let $z_i = (\lambda_i/\mu_i)/\sum_j(\lambda_j/\mu_j), 1 \leq i \leq d$. In a suitable scaled limit (the aforementioned Halfin-Whitt limit), the vector of total number of customers of various classes present in the system satisfies the controlled s.d.e.

$$dX(t) = b(X(t), u(t))dt + \Sigma dW(t),$$

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where the i-th component of $b(x,u)$ is $b_i(x,u) = -\theta\mu_i - \gamma_i(x_i - u_i) - \mu_i u_i$ and $\Sigma = diag[\sqrt{2\mu_1 z_1}, \cdots, \sqrt{2\mu_d z_d}]$. The parameter θ has the interpretation as the the excess capacity of the server pool in a suitable asymptotic sense. The action space is state-dependent and at x, is

$$U(x) = \{u \in \mathbb{R}^d : u \leq x, \sum_i u_i = (\sum_i x_i) \wedge 0\}.$$

The i-th component of the control, $u_i(t)$, will correspond to a scaled limit of the number of servers assigned to the class i at time t. The aim is to minimize the cost

$$\mathbb{E}\Big\{ \int_0^\infty e^{-\alpha t} c(X(t), u(t))dt \Big\}$$

for a discount factor $\alpha > 0$, where $c(x,u) = \sum_i (h_i + \gamma_i p_i)(x_i - u_i)$. Here h_i, p_i are resp. the holding cost and the abandonment penalty for class i.

1.4 Free boundary problems

There is a strong connection between stochastic optimal control problems and free boundary problems (see e.g.Bensoussan and Lions [12]). If we let the value function of the problem be $u(x,t)$ i.e., the infimum of J over all admissible controls, then an application of the dynamic programming principle will lead to the variational inequality or HJB equation. If we can show that the value function is convex and in $C^{2,1}(\mathbb{R}^n \times [0,T])$ and the boundary is smooth enough then it can be verified that the optimal control exists and has the following form: if the state process starts outside of Ω then the optimal control will make it jump to some point on the boundary $\partial\Omega$, thereafter control v acts only when the state process is on $\partial\Omega$, and is pushed it back into Ω. The optimal state process is thus a reflected diffusion in the set Ω, and the singular optimal control is like the local time of the reflected diffusion at the boundary $\partial\Omega$. Therefore, the free boundary problem related to the stochastic control therefore arises naturally.

1.5 Our main results

Under appropriate smoothness and growth conditions on the data, we prove the existence and uniqueness of polynomial growing, positive solution of the variational inequality associated with a

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multi-dimensional singular stochastic control problem with a convex running cost. The solution u of this variational inequality, which is the value function for the control problem, is shown to be in the class C^2. Moreover, our purpose is to prove that if for a fixed time t_0, the point X_0 is a point of density for the coincidence set, then in a neighborhood in space and time of (X_0, t_0) the free boundary is a surface of class C^1 in space and time and all the second derivatives of the solution are continuous up to the boundary.

This problem comes from the study of a linear stochastic control system. Let $(w(t), t > 0)$ be a standard Wiener process in $\mathbb{R}^n$, and the state of the system be described as

$$y(s) = x + v(s-t) + \int_t^s g(\lambda)d\lambda + \int_t^s \sigma(\lambda)dw(\lambda - t) \quad \text{for every } s \geq t, \tag{1.4}$$

where x is the initial state and $(v(s), s \geq 0)$ stands for the control which is a progressively measurable process with locally bounded variation, $g(\cdot)$ and $\sigma(\cdot)$ are given deterministic functions. The associated optimal control problem is to minimize an expected cost function defined by

$$J_{xt}(v) \;=\; \mathbb{E}\Big\{ \int_t^T f(y(s), s)ds + c(t)v(0) + \int_t^T c(s)d|v|(s-t)\Big\}, \tag{1.5}$$

where $f(\cdot,\cdot), c(\cdot)$ are given, $|v|$ denotes the variation of the process v and T is the finite horizon. Hence, the optimal cost function is

$$u(x,t) = \inf\{J_{xt}(v) : v\}, \text{ for every } x, t. \tag{1.6}$$

A formal application of the dynamic programming principle yields the complementary problem

$$\begin{cases} \max \Big\{Au - f, |D_i u| - c_i\Big\} = 0 \quad \text{in} \quad \mathbb{R}^n \times [0,T], \quad \forall i = 1, \cdots, n, \\ u(\cdot, T) \;=\; 0 \quad \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.7}$$

for the optimal cost (1.6), where

$$Au \;=\; -\frac{\partial u}{\partial t} - \frac{1}{2}\sum_{i,j=1}^n \left(\sum_{k=1}^n \sigma_{ik}(t)\sigma_{jk}(t)\right)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^n g_i(t)\frac{\partial u}{\partial x_i}, \tag{1.8}$$

and $|\cdot|$ denotes the absolute value of a real number.

It is clear that (1.7) can be regarded either as a variational inequality or as a free boundary problem. We are interested in the characteristics of an optimal policy of the control as well as a possible

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computation of that optimal strategy. On the other hand, we will see that the problem (1.7), commonly referred to as the monotone follower, see Karatzas [52], can be obtained as a limit case of a quasi-variational inequality.

This problem is motivated by our interest in studying the optimal control of a dissipative dynamical system under uncertainty. In the simplest model, one considers the automotive cruise of an aircraft under an uncertain wind condition. The equation (1.4) is the equation of motion, where $y(s)$ is the speed; $g(s)$ is the thrust force; the white noise term the dynamic force due to the shifting wind condition, and the formal derivative $\dot{v}$ represents the control in the form of a corrective thrust force. We wish to find an optimal control policy v over the flight time T so that given a finite amount of fuel for correction, the flight speed will deviate as little as possible to as desirable cruising speed at a minimum fuel cost. The system $(1.4)-(1.6)$ has another interesting interpretation in the context of optimal harvesting of randomly fluctuating resource, see Ludwig [64]. In that case, the equation stands for a controlled linear growth model for the size y of a population, say, in a fishery, where the terms g and $(\sigma\dot{w})$ are, respectively, the mean and fluctuating rates of migration, and $\dot{v}$ denotes the harvesting rate. For instance, in a finite horizon, we would like to determine the harvesting rate in order to maintain the population size as close as possible to an equilibrium size at a minimum cost.

In the multi-dimensional case [89], William *et al.* has been studied the problem with the elliptic operator.Moreover, in his book [68], Menaldi has also considered the optimal control of the stochastic different equations with jumps. In this dissertation, we study the general n-dimensional control problem with the parabolic operator. We will show that the optimal control exists under some conditions. The dynamic programming principle will be established.

An outline the main results of this thesis is as follows: Chapter 1 is a introduction of some recent aspects and developments in optimal stochastic control with a view towards applications. Chapter 2 presents the preliminary results that are necessary for the thesis. In Chapter 3, we recall a model problem which arise from different applications to show the particular features of optimal singular

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controls. We mention the characterization of the optimal cost function as the unique solution of the problem (1.7) in a certain sense. Also, we prove some preliminary results about the smoothness of the optimal cost function and studies certain other functions that approximate it. Chapter 4 devotes to prove the existence and uniqueness of solution of the penalized equation and the convergences to the optimal cost function, and lastly, some properties of regularity for the optimal cost, e.g. locally Lipschitzian derivative of u. The smoothness of the free boundary is proved in Chapter 5, with the main results in Theorem 5.13.

Beside our main results 'On a multi-dimensional singular stochastic control problem: the parabolic case', we also consider a backward parabolic problem, (see Nhat [86], [87]).

1.6 Backward parabolic problem

We consider an inverse time problem for a nonlinear parabolic equation in the form $u_t + Au(t) = f(t, u(t)), u(T) = \varphi$, where A is a positive self-adjoint unbounded operator and f is a Lipschitz function. As known, this problem is ill-posed. Using a quasi-reversibility method, we shall construct regularization solutions depending on a small parameter ϵ. We show that the regularized problem is well-posed and that their solution $u^\epsilon(t)$ converges on $[0, T]$ to the exact solution $u(t)$. These results extend the work by Dinh Nho Hao et al [42] to nonlinear ill-posed problems. Some numerical tests illustrate that the proposed method is feasible and effective. Let H be a Hilbert space. Let A be a self-adjoint operator defined on a subspace $D(A)$ of the vector space H such that $-A$ generates a compact contraction semi-group on H. We shall consider a final value problem of finding a $u : [0, T] \longrightarrow H$ such that

$$u_t + Au(t) = f(t, u(t)),\ 0 < t < T, \tag{1.9}$$

$$u(T) = \varphi, \tag{1.10}$$

where $\varphi \in H$ is a prescribed final value and $f : R \times H \longrightarrow H$ is a Lipschitz function. We can rewrite the above problem by the following integral equation (see, e.g., Balakrishnan [5], Chapter

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4)

$$u(t) = S(T-t)^{-1}\varphi - \int_t^T S(s-t)^{-1} f(s,u(s))ds, \tag{1.11}$$

where $S(t)$ is the semigroup (generated by $-A$) which is defined precisely later. As known, the nonlinear nonhomogeneous problem is severely ill-posed. In fact, the problem is extremely sensitive to measurement errors (see, e.g., Beck *et al.* [9]). The final data is usually the result of discretely experimental measurements and thus is patched into L^2-functions and that is subject to error. Hence, a solution corresponding to the data does not always exist, and in the case of existence, it does not depend continuously on the given data. This, of course, shows that a naturally numerical treatment is impossible. Hence, one has to resort to a regularization.

The problem has a long history. The linear homogeneous case $f = 0$ of this problem has been considered by many authors using different approaches. After the pioneering work by Lattes and Lions [59] in 1967, Miller [74], Payne [76] and many authors approximated the linear problem by perturbing the operator A. Their regularization methods called quasi-reversibility ((QR)method for short) are effective to the homogeneous problem but the nonhomogeneous and the nonlinear cases have not been studied completely. The main idea of the method is adding a "corrector" into the main equation. In fact, they considered the problem

$$u_t + Au - \epsilon A^* Au = 0, \quad t \in [0,T],\ u(T) = \varphi.$$

The stability magnitude of the method are of order $e^{c\epsilon^{-1}}$. In [2], [39], [81] the problem is approximated with

$$u_t + Au + \epsilon A u_t = 0, \quad t \in [0,T],\ u(T) = \varphi. \tag{1.12}$$

Ames and Hughes [4] gave a survey about an association between the operator-theoretic methods and the QR method to treat the abstract Cauchy problem

$$\frac{du}{dt} = Au,\ u(T) = \chi, \quad 0 < t < T.$$

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The authors considered the problem in both the Hilbert space and in the Banach space. They also gave many structural stability results. Recently, using the QR method, Yongzhong Huang and Quan Zheng, in [48], considered the problem (1.9) in an abstract setting, i.e., $-A$ is the generator of an analytic semigroup in a Banach space. In 1983, Showalter presented a different method called the quasiboundary value (QBV) method to regularize that linear homogeneous problem which gave a stability estimate better than the one of discussed methods. The main idea of the method is of adding an appropriate "corrector" into the final data. Using this method, Clark-Oppenheimer, in [27], and Denche-Bessila, recently in [29], regularized the backward problem by replacing the final condition by

$$u(T) + \epsilon u(0) = \varphi$$

and

$$u(T) - \epsilon u'(0) = \varphi,$$

respectively.

Recently, the improve results for homogeneous ill-posed problem has also been given in [42] by Dinh Nho Hao and his coauthors.

Although we have many work on the linear homogeneous case of the backward problem, the literature on the linear nonhomogeneous case and the nonlinear case of the problem are quite scarce. In Trong and Tuan [85], the authors used the QR method and the eigenvalue expansion method to regularize a 1-D linear nonhomogeneous backward problem. Recently, in Quan and Trong [80], the methods of integral equations and of Fourier transform have been used to solve a 1-D nonlinear problem on R.

As far as we known up to now, we can only find some rarely papers which studied the nonlinear backward problem by using quasi-reversibility, such as Long and Dinh [63]. In fact, in [63] gave the

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following problem

$$v'_\beta(t) + A_\beta v_\beta(t) = e^{-(1-t)\beta A A_\beta} f(v_\beta)$$

$$v_\beta(1) = \varphi$$

where $A_\beta = A(I + \beta A)^{-1}$. However, they estimated some error between the exact solution and the approximate solution on the small interval of time. Moreover, the stability of magnitude of that problem is very large, which is $e^{\frac{c}{\epsilon}}$. Consequently, the quasi-reversibility method given in Long and Ngoc [63], is not effective to regularize the backward parabolic problem with the large time.

It is believed that the QBV method gives the stability result better than the other QR method do. In this dissertation, we shall use a modified quasi-reversibility method to regularize the problem and to improve the stability result of this method. We shall prove that this method gives the same stability magnitude order as the one in the case of QBV method. And especially, the new method is useful to consider nonlinear problems. The problem (1.11) can be approximated by the approximate problem

$$\frac{d}{dt}u^\epsilon(t) + A_\epsilon u^\epsilon(t) = B(\epsilon, t) f(t, u^\epsilon(t)), \qquad t \in [0, T], \tag{1.13}$$

$$u^\epsilon(T) = \varphi \tag{1.14}$$

where $A_\epsilon, B(\epsilon, t)$ will be defined later on.

The remainder of the thesis is Chapter 6. In section 6.1, we shall prove that (1.13) – (1.14) are well-posed. Then, in section 6.2, we shall show that u^ϵ converges in $C([0, T]; H)$ to the exact solution. Error estimates are then given.

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CHAPTER 2 PRELIMINARIES

In this chapter, we present some fundamental results on dynamic programming principle (see Flemming [36]), regularity of parabolic problems (see Bensoussan [12]) as well as the free boundary problem (see Friedman [38]).

2.1 Dynamic programming principle

Define a value function by $u(x,t) = \inf_{u(\cdot)\in\mathcal{U}(x,t)} J(x,t;u)$. For any initial condition $(x,t) \in \bar{Q}$ and $r \in [t, t_1]$,

$$u(x,t) = \inf_{u(\cdot)\in\mathcal{U}(x,t)} \Big[\int_t^{r\wedge\tau} L(s, x(s), u(s))ds + g(\tau, x(\tau))\chi_{\tau<r} + u(r, x(r))\chi_{\tau<r} \Big]. \tag{2.1}$$

The above identity is called the dynamic programming principle. Let $0 < h \leq t_1 - t$, and take $r = t+h$ in the dynamic programming principle (2.1). Subtract $u(x,t)$ from both sides of (2.1) and then divide by h. This yields

$$\begin{aligned} \inf_{u(\cdot)\in\mathcal{U}(x,t)} \Big\{ \frac{1}{h} \int_t^{(t+h)\wedge\tau} & L(s, x(s), u(s))ds + \frac{1}{h} g(\tau, x(\tau))\chi_{\tau<t+h} \\ & + \frac{1}{h}[u(t+h, x(t+h))\chi_{t+h<\tau} - u(t,x)] \Big\} = 0. \end{aligned} \tag{2.2}$$

For every $(t,x) \in Q$ and $v \in \mathcal{U}$ there exists $u(\cdot) \in \mathcal{U}(x,t)$ such that $v = \lim_{s\downarrow t} u(s)$. If we formally let $h \downarrow 0$ in (2.2) we obtain

$$\frac{\partial}{\partial t}V(x,t) + \inf_{v\in U}\{L(t,x,v) + f(t,x,v)\cdot D_x V(x,t)\} = 0. \tag{2.3}$$

This is a nonlinear partial differential equation of first order, which we refer to as the dynamic programming equation. In (2.3), $D_x V$ denotes the gradient of $V(t,\cdot)$. It is convenient to rewrite (2.3) as

$$-\frac{\partial}{\partial t}V(x,t) + H(t,x,D_x V(t,x)) = 0, \tag{2.4}$$

where $(t,x,p) \in \bar{Q} \times \mathbb{R}^n, H(t,x,p) = \sup_{v\in U}\{-p\cdot f(t,x,v) - L(t,x,v)\}$. We call this function the Hamiltonian The dynamic programming equation (2.4) is also called a Hamilton-Jacobi-Bellman PDE.

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We will also introduce an important moment inequality as follow:

Theorem 2.1. *(see [67], page 39) Let $p \geq 2$. Let $g \in \mathcal{M}^2([0,T];\mathbb{R}^{d\times m})$ such that*

$$\mathbb{E}\Big\{\int_0^T |g(s)|^p ds\Big\} < \infty.$$

Then

$$\mathbb{E}\Big|\int_0^T g(s)dB(s)\Big|^p \leq \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} T^{\frac{p-2}{2}} \mathbb{E}\Big\{\int_0^T |g(s)|^p ds\Big\}.$$

In particular, for $p = 2$, there is equality.

2.2 Parabolic P.D.E.'s of second order in $\mathbb{R}^n \times]0,T[$

2.2.1 Regularity with respect to the space variables

We denote by $W^{2,1,p}(Q)$ the space of the functions u such that $u, \dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x_i}, \dfrac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(Q), 1 \leq p \leq \infty$, equipped with the natural Banach or Hilbert space norm if $p = 2$; in the notation $W^{2,1,p}(Q)$, the "1" refers to the number of derivatives with respect to t which are in L^p and the "2" refers to the number of derivatives with respect to x; if $p = 2$ we write $W^{2,1}(Q)$. We denote by $W^{2,1,p}_{loc}(Q)$ the space of the function u such that $\varphi \in \mathcal{D}(Q)$ we have $\varphi u \in W^{2,1,p}(Q)$. We take functions $a_{ij}(t), a_i, a_0$ on $\mathbb{R}^n \times]0,T[$ which satisfy

$$a_{ij} = a_{ji}, a_{ij}, a_i, a_0 \in C^1(\mathbb{R}^n \times]0,T[). \tag{2.5}$$

Let $v \in L^p_{loc}(Q)$. We denote by Lv the following distribution on Q:

$$< Lv, \psi >= \int_Q v\Big(\frac{\partial \psi}{\partial t} - \sum_{ij} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial \psi}{\partial x_i}) - \sum_i \frac{\partial}{\partial x_i}(a_i \psi) + a_0 \psi\Big) dxdt$$

where $\psi \in \mathcal{D}(Q)$.

Theorem 2.2. *(see [12], page 131) Suppose that the assumptions* (2.5) *hold. Let $u \in L^p_{loc}(Q)$ be such that*

$$Lu = -u_t + Au = f \in L^p_{loc}(Q), \quad \textit{then} \quad u \in W^{2,1,p}_{loc}(Q).$$

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If G is a bounded open set $\subset Q$ and G' is an open set such that $G' \subset G$, then we have

$$\|u\|_{W^{2,1,p}(G')} \leq C(|f|_{L^p(G)} + |u|_{L^p(G)}),$$

the constant C being dependent on the bounds on the coefficients of L on G, as well as on G and G'.

Remark 2.1. *Under the assumptions (2.5), and if we also have $\dfrac{\partial^2 a_{ij}}{\partial t \partial x_i}, \dfrac{\partial^2 a_{ij}}{\partial x_l \partial x_i} \in L^p_{loc}(Q), f \in L^p_{loc}(Q), \dfrac{\partial f}{\partial t}, \dfrac{\partial f}{\partial x_i} \in L^p_{loc}(Q)$, $u \in L^p_{loc}(Q)$ and $Lu = f$, then we have $u \in W^{3,1,p}_{loc}(Q), \dfrac{\partial u}{\partial t} \in W^{2,1,p}_{loc}(Q)$. In particular, if $p > n$, then $u \in C^{2,1}(Q)$.*

2.2.2 Unbounded coefficients

We write

$$A(t) = -\sum \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial}{\partial x_j}) + \sum a_{ij}(x,t)\frac{\partial}{\partial x_i} + a_0(x,t)I.$$

We adopt the assumptions

$$a_{ij} = a_{ji}, \quad |a_{ij}(x,t)| \leq C, \quad \sum a_{ij}(x,t)\xi_i\xi_j \geq \alpha \sum \xi_i^2, \quad \alpha > 0, \tag{2.6}$$

$$|\frac{\partial a_i}{\partial x_j}(x,t)| \leq cm(x), \quad -\frac{1}{2}\frac{\partial a_i}{\partial x_i} \geq \beta_0 m - c_1,$$

β_0, c, c_1 suitable constants, (2.7)

$m(x)$ is a positive continuous function in $\mathbb{R}^n$ such that $\sup_{0 \leq t \leq 1} m(tx) \leq cm(x)$,

$$a_0 \in L^\infty, \text{ and } a_0(x,t) \geq \beta > 0. \tag{2.8}$$

We introduce

$$\pi(x) = (1+x^2)^{-s}, \quad s \geq 0 \text{ fixed arbitrary},$$
$$L^2_\pi = \{v | \pi^{\frac{1}{2}} v \in L^2(\mathbb{R}^n)\}; \quad H^1_\pi = \left\{v \in L^2_\pi | \frac{\partial v}{\partial x_i} \in L^2_\pi\right\}. \tag{2.9}$$

We put

$$|v|^2_\pi = \int \pi v^2 dx,$$
$$\|v\|^2_\pi = |v|^2_\pi + \sum_i |\frac{\partial v}{\partial x_i}|^2_\pi, \tag{2.10}$$

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which defined Hilbert norms on L^2_π and H^1_π respectively.

We introduce

$$F = \{v | v \in H^1_\pi,\ v\sqrt{m} \in L^2_\pi\}; \text{equipped with the norm}$$
$$\|v\|_F = \left(\|v\|^1_{H^2_\pi} + |m^2 v|^2_{L^2_\pi}\right)^{\frac{1}{2}}. \tag{2.11}$$

Theorem 2.3. *(see [12], page 134) Suppose the assumptions* (2.6), (2.7), (2.8) *hold. Let* $f \in L^2(0,T;L^2_\pi)$ *and* $\bar{u} \in L^2_\pi$. *Then there exists a unique function* u *such that*

$$u \in L^2(0,T;F),$$
$$-\frac{\partial u}{\partial t} + A(t)u = f, \tag{2.12}$$
$$u(T) = \bar{u}.$$

Theorem 2.4. *(see [12], page 139) Suppose that the conditions of Theorem 2.3 hold and that in addition we have*

$$|a_{ij}(x,t)|(1+m(x)) \le C,$$
$$\Big|\frac{\partial a_{ij}}{\partial x_k}(x,t)\Big|(1+|x|m(x)) \le C. \tag{2.13}$$

For $f \in L^2(0,T;L^2_\pi)$ *and* $\bar{u} \in H^1_\pi$, *the solution* u *given in Theorem 2.3 satisfies*

$$-\sum \frac{\partial u}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j}) \in L^2(0,T;L^2_\pi), \quad u \in L^\infty(0,T;F).$$

Remark 2.2. *If* $f \in L^p(0,T;L^p_\pi(\mathbb{R}^n))$ *then* $u \in L^p(0,T;L^p_\pi(\mathbb{R}^n))$, *so that, using Theorem 2.2, we have* $u \in W^{2,1,p}_{loc}(\mathbb{R}^n \times]0,T[)$. *If* $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i} \in L^p_{loc}(Q)$ *then* $u \in W^{3,1,p}_{loc}(\mathbb{R}^n \times]0,T[)$ *and* $\frac{\partial u}{\partial t} \in W^{2,1,p}_{loc}(Q)$, *with implies, if* $p > n, u \in C^{2,1}(\mathbb{R}^n \times]0,T[)$.

We then adopt the assumptions

$$\frac{\partial f}{\partial t} \in L^2(0,T;F'), \quad A(T)\bar{u} - f(T) \in L^2_\pi \tag{2.14}$$

$$\frac{\partial a_{ij}}{\partial t}, \frac{\partial a_0}{\partial t}, \frac{\partial a_j}{\partial t} \in L^\infty(\mathbb{R}^n \times (0,T)) \tag{2.15}$$

Theorem 2.5. *(see [12], page 142) Under the assumptions of Theorem 2.3 with* (2.14), (2.15), *the solution* u *obtained in Theorem 2.3 satisfies*

$$\frac{\partial u}{\partial t} \in L^2(0,T;F). \tag{2.16}$$

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Corollary 2.1. *(see [12], page 143) Under the condition of Theorems 2.4 and 2.5, we have*

$$u \in L^\infty(0,T;F), \frac{\partial u}{\partial t} \in L^2(0,T;F), -\sum \frac{\partial u}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j}) \in L^2(0,T;L^2_\pi), \tag{2.17}$$

$$\sum a_{ij}(x,t)\frac{\partial u}{\partial x_i} \in L^2(0,T;L^2_\pi). \tag{2.18}$$

We adopt the assumptions

$$m(x) = 1, \tag{2.19}$$

$$\frac{\partial a_{ij}}{\partial x_k}, \frac{\partial a_0}{\partial x_k} \in L^\infty(\mathbb{R}^n \times (0,T))\ \forall k, \tag{2.20}$$

$$\bar{u} \in H^1_\pi. \tag{2.21}$$

We then have the following theorem:

Theorem 2.6. *(see [12], page 143) Under the assumptions of Theorem 2.3 and with* (2.19), (2.20), (2.21), *we have:*

$$u, \quad \frac{\partial u}{\partial x_k} \in L^2(0,T;H^1_\pi)\ \forall k, \tag{2.22}$$

where u is the solution obtained in Theorem 2.3.

2.2.3 Bounded coefficients

We consider the family of differential operators introduced in the preceding section. The coefficients now satisfy the following assumptions

$$\begin{aligned} & a_{ij}(x,t) = a_{ji}(x,t),\ a_{ij},\ a_i,\ a_0 \in L^\infty(\mathbb{R}^n \times]0,T[), \\ & \sum_{ij} a_{ij}(x,t)\xi_i\xi_j \geq \alpha \sum_i \xi_i^2, \quad \alpha > 0, \\ & \ a_0(x,t) \geq \beta > 0. \end{aligned} \tag{2.23}$$

We denote by $W^{m,p,\mu}$ the space of functions $u(x)$ such that the quantity

$$|u|_{m,p,\mu} = \Big(\sum_{k \leq m} \int_{\mathbb{R}^n} \exp(-\mu|x|)|D^k u(x)|^p\Big)^{1/p} < \infty. \tag{2.24}$$

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Equipped with the norm (2.24), $W^{m,p,\mu}$ is a Banach space. We put $H^{m,\mu} = W^{m,2,\mu}$, $H_\mu = H^{0,\mu}$, $V_\mu = H^{1,\mu}$, and we define a continuous bilinear form on V_μ by means of the formula

$$\begin{aligned} a(u,v) = &\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} a_{ij} m_\mu \frac{\partial u}{\partial x_j} m_\mu \frac{\partial u}{\partial x_i} dx + \\ &+ \sum_{i=1}^{n} \int_{\mathbb{R}^n} (a_i - 2\mu \sum_j a_{ij} \frac{x_j}{|x|}) m_\mu \frac{\partial u}{\partial x_i} m_\mu v dx \\ &+ \int_{\mathbb{R}^n} a_0 m_\mu u m_\mu v dx \end{aligned} \tag{2.25}$$

in which we have put

$$m_\mu(x) = \exp(-\mu|x|).$$

Theorem 2.7. *(see [12], page 144) Suppose that* (2.23) *holds. Let $f \in L^2(V'_\mu)$ and $u \in H'_\mu$; then there exists one and only one element u such that $u \in L^2(V_\mu), \dfrac{\partial u}{\partial t} \in L^2(V'_\mu)$, satisfying*

$$\begin{aligned} &-(\frac{du}{dt}, v) + a(t; u(t), v) = (f(t), v) \quad a.e.\ t \in]0,T[, \quad \forall v \in V'_\mu, \\ &u(T) = \bar{u}. \end{aligned} \tag{2.26}$$

Theorem 2.8. *(see [12], page 145) Under the assumptions of Theorem 2.7 and $\left|\dfrac{\partial a_{ij}}{\partial t}\right| \leq C$, for $f \in L^2(H_\mu)$ and $u \in V_\mu$) the solution $u(t)$ of* (2.26) *belongs to $L^\infty(V_\mu)$ and $\dfrac{du}{dt} \in L^2(H_\mu)$.*

It then follows Theorem 2.8 that $u \in L^2(0,T; D(A(t)))$ and that

$$\begin{aligned} &-\frac{\partial u}{\partial t} + A(t)u = f \quad \text{a.e. } x, t \\ &u(T) = \bar{u}. \end{aligned} \tag{2.27}$$

Theorem 2.9. *(see [12], page 145) Suppose that* (2.23) *holds and that $\dfrac{\partial a_{ij}}{\partial x_k}, \dfrac{\partial a_{ij}}{\partial t}$ are bounded. Let $f \in L^p(0,T; V^{0,p,\mu}) \cap L^2(0,T; H_\mu)$ and $u = 0$; then the solution of* (2.27) *satisfies the regularity property $u \in L^p(0,T; W^{2,p,\mu})$ and $\dfrac{\partial u}{\partial t} \in L^p(0,T; W^{0,p,\mu})$. Additionally, we have the estimate*

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^p(0,T;W^{0,p,\mu})} + \|u\|_{L^p(0,T;W^{2,p,\mu})} \leq C\|f\|_{L^p(0,T;W^{0,p,\mu})}. \tag{2.28}$$

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2.3 General properties of the free boundary

For parabolic operators one can solve the first initial boundary value problem

$$u_t + Au = f \text{ in } Q,$$
$$u = g \text{ on } \partial_p Q. \tag{2.29}$$

With appropriate assumptions, (2.29) can be written in the weak form

$$(u_t,\, v-u) + a(t; u, v-u) = (f, v-u) \quad \text{for a.e. } t \in (0,T),$$
$$\forall v \in H^1(Q), \quad v = g \text{ on } \partial_p Q, \tag{2.30}$$

where $(v,w) = \int\limits_{\Omega} vw dx$. We will study one type of parabolic variational inequalities. Find u satisfying

$$\begin{cases} u_t + Au \geq f, \\ u \geq \phi, \\ (u_t + Au - f)(u-\phi) = 0, \\ u = g \text{ on } \partial_p Q. \end{cases} \tag{2.31}$$

Let us consider solution of (2.31) in the special case $A = -\Delta, \phi = 0$. We find that $u_t - \Delta u = f$ if $u > 0$. Also, clearly $u_t - \Delta u = 0 > f$ if $u = 0$. The function u takes on $\partial_p Q$ the boundary value ψ, where

$$\psi(x,t) \;=\; \begin{cases} \int\limits_0^t g(x,\tau)d\tau & \text{if} \quad a \in \Gamma_0,\ t>0, \\ 0 & \text{if} \quad t = 0 \text{ or if } |x| = R, \end{cases} \tag{2.32}$$

with assumption that $g \in C^{2+\alpha}(\Gamma_0 \times [0,T]), \quad g > 0$. Introducing the convex set

$$K = \{v \in H^1(Q), v \geq 0 \text{ a.e. in } Q, v = \psi \text{ on } \partial_p Q\}. \tag{2.33}$$

If u is a solution of the variational inequality

$$u \in K,$$
$$\int\limits_{\Omega} u_t(v-u)dx + \int\limits_{\Omega} \nabla u \nabla v dx \geq \int\limits_{\Omega} f(v-u)dx \quad \text{for a.e. } t \in (0,T) \quad \forall v \in K. \tag{2.34}$$

Theorem 2.10. *(see [38], page 84) The solution u of the problem* (2.34) *satisfies*

$$0 \leq u_t \leq C, \quad C < \infty$$

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and

$$D_x u, D_x^2 u, D_t u \text{ belong to } L^\infty((0,T); L^p(\Omega)).$$

The set $N = \{x \in \Omega : u(x) > 0\}$ is called the *noncoincidence set* and the set $\Lambda = \{x \in \Omega : u(x) = 0\}$ known as the *coincidence set.* the boundary of the noncoincidence set in Ω

$$\Gamma = \partial N \cap \Omega$$

is called the *free boundary.* We introduce two basic facts for the obstacle problem:

- The free boundary has measure zero;
- If $y \in \Gamma$ then $\liminf_{x \to y,\ x \in N} D_{ii}(u(x) - \phi(x)) \geq 0$, where ϕ is the obstacle, and i is any direction.

Definition 2.1. *(see [38], page 162) For any bounded set S, the minimum diameter of $MD(S)$ is the infimum of distances between pairs Π_1, Π_2 of parallel planes such that S is contained in the strip determined by Π_1, Π_2. Define the thickness of Λ at the free boundary point (x_0, t_0) by*

$$\delta_r(\Lambda) = \frac{MD(\Lambda_{t_0} \cap B_r(x_0))}{r}.$$

Theorem 2.11. *(see [38], page 235) Let (x_0, t_0) be a free boundary point; $t_0 > 0$. Then there is exists a positive nondecreasing function $\sigma(r)(0 < r < r_0)$ with $\sigma(0+) = 0$ such that if, for some $0 < r < r_0$,*

$$\delta_r(\Lambda) > \sigma(r),$$

then there exist a neighborhood V of (x_0, t_0) such that $V \cap \Gamma$ can be represented in the form

$$x_i = k(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t)$$

with $k \in C^1$, and all the second derivatives of u $(D_x^2 u, D_x D_t u, D_t^2 u)$ are continuous in $(N \cup \Gamma) \cap V$.

Corollary 2.2. *(see [38], page 235) If*

$$\limsup_{r \to 0} \frac{|\Lambda_{t_0} \cap B_r(x_0))|}{|B_r|} > 0,$$

then the assertions of Theorem 2.11 are valid.

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CHAPTER 3 FORMULATION OF STOCHASTIC OPTIMAL CONTROL PROBLEM AND CHARACTERIZATION OF THE OPTIMAL COST

3.1 The Formulation of Expected Cost Function

Let $(\mathbf{\Omega}, \mathcal{F}, P)$ be a probability space, where $(\mathcal{F}(t), t \geq 0)$ be a filtration satisfying the usual conditions with respect to $(w(t), t \geq 0)$, i.e., $(\mathcal{F}(t), t \geq 0)$ is an increasing right continuous family of completed σ-subalgebras of $\mathcal{F}$ and $(w(t), t \geq 0)$ is a martingale with respect to $(\mathcal{F}(t), t \geq 0)$.

Denote by V be the admissible set of singular control $v(\cdot)$ which are progressively measurable random processes from $[0, \infty)$ into $\mathbb{R}^n$, right continuous having left limits (cad-lag), nonnegative and increasing, i.e,

$$v_i(0) \geq 0, \quad v_i(s) - v_i(t) \geq 0 \text{ for every } s \geq t \geq 0, \quad \forall i = 1, \cdots, n. \tag{3.1}$$

Let $y(t) = (y_1(t), \cdots, y_n(t))$ denote the state at time t of a controlled system governed for $t \geq 0$ by the following Ito's equations

$$y_i(s) = x_i + v_i(s-t) + \int_t^s g_i(\lambda)d\lambda + \sum_{j=1}^n \int_t^s \sigma_{ij}(\lambda)dw_j(\lambda - t), \quad s \geq t, \tag{3.2}$$

for $i = 1, \cdots, n$, where $x = (x_1, \cdots, x_n)$ is the initial state and $v = (v_1, \cdots, v_n)$ is the control vector, $g = (g_1, \cdots, g_n)$ is the drift vector, $\sigma = [\sigma_{ij}]$, $i, j = 1, \cdots, n$ is the diffusion matrix, and $w(t) = (w_1(t), \cdots, w_n(t))$ is a standard Wiener process in $\mathbb{R}^n$. The expected cost takes the form

$$\begin{aligned} J_{xt}(v) = \; & \mathbb{E}\Big\{ \int_t^T f(y_{xt}(s), s) \exp\Big(- \int_t^s \alpha(\lambda)d\lambda\Big) ds \\ + \; & \sum_{i=1}^n c_i \int_t^T \exp\Big(- \int_t^s \alpha(\lambda)d\lambda\Big) dv_i(s-t)\Big\}, \end{aligned} \tag{3.3}$$

where y_{xt} is used in place of y to emphasize the dependence on the initial state x and time t, $f(\cdot,\cdot)$ is the given running cost from $\mathbb{R}^n \times [0,T]$ into $\mathbb{R}$, $c_i \geq 0$ is given and T is the finite horizon, $\alpha(t) > \alpha_0 > 0$ is discount factor, where $\alpha_0 = \min\limits_{t \in [0,T]} \alpha(t)$. The optimal cost is given by

$$u(x,t) = \inf\{J_{xt}(v) : v \in V\}, \; x \in \mathbb{R}^n, \; t \in [0,T]. \tag{3.4}$$

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The aim is to characterize the value function $u(x,t)$ and to obtain an optimal control, i.e., finding

$$\hat{v} \text{ in } V \text{ such that } u(x,t) = J_{xt}(\hat{v}). \tag{3.5}$$

For each $\epsilon > 0$, let V^ϵ denote the set of all controls $v \in V$ such that v is Lipschitz continuous with probability one and

$$0 \le \frac{dv_i}{dt}(t) \le \frac{1}{\epsilon} \quad \text{a.e. for } t \ge 0,\ i = 1, \cdots, n \quad \text{a.s.} \tag{3.6}$$

The corresponding optimal cost function $\hat{u}^\epsilon$ is given by

$$\hat{u}^\epsilon(x,t) = \inf\{J_{xt}(v) : v \in V^\epsilon\},\ x \in \mathbb{R}^n,\ t \in [0,T]. \tag{3.7}$$

Assume that the following conditions hold:

$$g_i, \sigma_{ij} \text{ are Lipschitz functions for any } i,j = 1, \cdots, n. \tag{3.8}$$

3.2 Hamilton-Jacobi-Bellman Equation

Let A be the nonlinear parabolic operator

$$Au = -\frac{\partial u}{\partial t} - \frac{1}{2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}\sigma_{ik}(t)\sigma_{jk}(t)\right)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{n} g_i(t)\frac{\partial u}{\partial x_i} + \alpha(t)u, \tag{3.9}$$

where $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \in [0,T]$. Then for problem (3.7) an application of the dynamic programming principle yields the Hamilton-Jacobi-Bellman equation for the value function $\hat{u}^\epsilon$:

$$\begin{cases} A\hat{u}^\epsilon + \dfrac{1}{\epsilon}\displaystyle\sum_{i=1}^{n}\left(\frac{\partial \hat{u}^\epsilon}{\partial x_i} + c_i\right)^- = f,\ x \in \mathbb{R}^n,\ t \in [0,T], \\ \hat{u}^\epsilon(\cdot, T) = 0. \end{cases} \tag{3.10}$$

(For any $t \in \mathbb{R}$, let $t^+ = \max\{t, 0\}$ and $t^- = \max\{-t, 0\}$ denote the positive and negative part of t respectively). As $\epsilon \to 0_+$, we can deduce from (3.10) that the solution u of the original problem (3.4) satisfies the variational inequality

$$\begin{cases} Au \le f, \quad \nabla u + c_i \ge 0, \\ (Au - f)\displaystyle\prod_{i=1}^{n}\left(\frac{\partial u}{\partial x_i} + c_i\right) = 0, \\ u(\cdot, T) = 0 \quad \text{in} \quad \mathbb{R}^n \times [0,T], \end{cases} \tag{3.11}$$

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where $a_{ij}(t)=\dfrac{1}{2}\sum\limits_{k=1}^{n}\sigma_{ik}(t)\sigma_{jk}(t)$. We take functions $a_{ij}(t)$ satisfy

$$\sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j \geq \alpha\sum_{i=1}^{n}\xi_i^2,\ \alpha>0,\ \forall\xi_1,\cdots,\xi_n\in\mathbb{R}. \tag{3.12}$$

3.3 Preliminary Results On The Smoothness Of u

First, we will prove some priori estimates for the optimal cost (3.4).

Let us summarize the technical assumptions as follows:

$$\begin{cases} T \text{ is a positive constant, } c_i \text{ is nonnegative constants,}\\ \alpha(t) \text{ is a nonnegative continuous function on } [0,T],\\ f\in C^3(\mathbb{R}^n\times[0,T]),\ f \text{ is convex in } x\\ \text{and there exist constants } m\geq 1, 0<k\leq K \text{ satisfying}\\ (i)\quad k|x^+|^m-K\leq f(x,t)\leq K(1+|x|^m),\\ (ii)\quad |f(x,t)-f(x',t)|\leq K(1+|x|^{m-1}+|x'|^{m-1})|x-x'|,\\ (iii)\quad |f(x,t)-f(x,t')|\leq K(1+|x|^m)|t-t'|,\\ (iv)\quad 0\leq \dfrac{\partial^2 f}{\partial x_i^2}(x,t)\leq K(1+|x|^q),\quad q=(m-2)^+,\quad i=1,\cdots,n,\\ \text{for every } x,x',t,t'. \end{cases} \tag{3.13}$$

Throughout this paper, we use K to denote a generic positive constant which may differ from line to line. Some estimates for the optimal cost function are given in the following lemmas.

Lemma 3.1. *Suppose that* (3.8) *and* (3.13) *hold. Define* $V_{xt}=\{v\in V: J_{xt}(v)\leq J_{xt}(0)\}$. *Clearly we have* $u(x,t)=inf\,\{J_{xt}(v): v\in V_{xt}\},\ x\in\mathbb{R}^n,\ t\in[0,T]$. *Moreover, there are positive constants* K,k *independent of* x,t *such that*

$$\begin{cases} (i)\quad J_{xt}(v)\leq K(1+|x|^m),\quad \forall v\in V_{xt},\\ (ii)\quad \mathbb{E}\{v(T-t)\}\leq K(1+|x|^m) \textit{ for any } v\in V_{xt},\\ (iii)\quad \mathbb{E}\Big\{\displaystyle\int_t^T|y_{xt}(s)|^m ds\Big\}\leq K(1+|x|^m),\quad \forall v\in V_{xt},\\ (iv)\quad \mathbb{E}\Big\{\displaystyle\int_t^T|y_{xt}(s)^+|^m ds\Big\}\geq k|x|^m-K,\quad \forall v\in V_{xt}. \end{cases} \tag{3.14}$$

Proof. Proof of item (3.14-i).

Consider the particular admissible control vanishing everywhere, i.e. $v=0$, we have

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$$|y_{xt}^0(s)| \le |x| + \int_t^s |g(\lambda)|d\lambda + \sum_{i,j=1}^n \Big| \int_t^s \sigma_{ij}(\lambda) dw_j(\lambda - t)\Big|$$

implying

$$\mathbb{E}\Big\{|y_{xt}^0(s)|^m\Big\} \le C\Big(|x|^m + \int_t^s |g(\lambda)|^m d\lambda + \sum_{i,j=1}^n \mathbb{E}\Big\{\Big|\int_t^s \sigma_{ij}(\lambda)dw_j(\lambda - t)\Big|^m\Big\}\Big)$$

where C depends on m. In view of Theorem 2.1 we have an upper bound

$$\begin{aligned}
\mathbb{E}\Big\{\Big|\int_t^s \sigma_{ij}(\lambda)dw_j(\lambda - t)\Big|^m\Big\} &= \mathbb{E}\Big\{\Big|\int_0^{s-t} \sigma_{ij}(\lambda + t)dw_j(\lambda)\Big|^m\Big\} \\
&\le \mathbb{E}\Big\{\Big|\int_0^{T-t} \sigma_{ij}(\lambda + t)dw_j(\lambda)\Big|^m\Big\} \\
&\le \Big(\frac{m(m-1)}{2}\Big)^{\frac{m}{2}} T^{\frac{p-2}{2}} \mathbb{E}\Big\{\int_0^{T-t} \Big|\sigma_{ij}(\lambda + t)\Big|^m d\lambda\Big\}.
\end{aligned} \tag{3.15}$$

Since g and σ are measurable and bounded then we have

$$\mathbb{E}\{|y_{xt}^0(s)|^m\} \le K(1 + |x|^m) \tag{3.16}$$

since $t, s \le T < \infty$. By the formula of the expected cost in (3.3) and by using (3.13-i) we have

$$\begin{aligned}
J_{xt}(0) = \mathbb{E}\Big\{\int_t^T f(y_{xt}^0(s), s) \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} &\le K\mathbb{E}\{1 + |y_{xt}^0(s)|^m\} \\
&\le K(1 + |x|^m)
\end{aligned} \tag{3.17}$$

for some constant $K > 0$. Using (3.4), we obtain

$$J_{xt}(v) \le J_{xt}(0) \le K(1 + |x|^m), \quad \forall x \in \mathbb{R}^n,\ t \in [0, T], \forall v \in V_{xt}.$$

Proof of item (3.14-ii).

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Since $\alpha(t)$ is bounded, and $f(x,t) \geq -K$, it is easy to see that for $v \in V_{xt}$,

$$\begin{aligned}
\mathbb{E}\Big\{v(T-t)\Big\} &= \mathbb{E}\Big\{v(0) + \int_t^T dv(s-t)\Big\} \\
&\leq \mathbb{E}\Big\{\kappa_1\Big(\sum_{i=1}^n c_i v(0) + \int_t^T \sum_{i=1}^n c_i \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)dv(s-t)\Big)\Big\} \\
&\leq \kappa_1 J_{xt}(v) + \kappa_2 \\
&\leq K(1+|x|^m) \text{ for some } \kappa_1, \kappa_2, K > 0.
\end{aligned} \tag{3.18}$$

Proof of item (3.14-iii).

Using (3.3) and the inequality $k|x^+|^m - K \leq f(x,t)$, we have

$$\begin{aligned}
k\mathbb{E}\Big\{\int_t^T \Big|[y_{xt}(s)]^+\Big|^m \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} - K &\leq \mathbb{E}\Big\{\int_t^T f(y_{xt}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} \\
&\leq J_{xt}(v) \leq J_{xt}(0) \\
&\leq K(1+|x|^m),\ \forall x \in \mathbb{R}^n,\ t \in [0,T],\ \forall v \in V_{xt}.
\end{aligned}$$

Therefore for some other $K > 0$ we obtain

$$\mathbb{E}\Big\{\int_t^T \Big|[y_{xt}(s)]^+\Big|^m ds\Big\} \leq K(1+|x|^m). \tag{3.19}$$

Proof of item (3.14-iv).

It follows form (3.16) that

$$\begin{aligned}
\mathbb{E}\Big\{\int_t^T \Big|y^0_{xt}(s)\Big|^m \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} &\leq \mathbb{E}\Big\{\int_t^T \Big|y^0_{xt}(s)\Big|^m ds\Big\} \\
&\leq K(1+|x|^m), \quad \forall x \in \mathbb{R}^n,\ t \in [0,T].
\end{aligned} \tag{3.20}$$

By (3.2), $y^0_{xt} = y_{xt} - v$, $v \geq 0$, we deduce that

$$\begin{aligned}
|y_{xt}(s)| \leq&\quad |[y_{xt}(s)]^+| + |y_{xt}(s) - v| \\
\leq&\quad |[y_{xt}(s)]^+| + |y^0_{xt}(s)|.
\end{aligned} \tag{3.21}$$

Using (3.19), (3.20) then we obtain

$$\mathbb{E}\Big\{\int_t^T \Big|y_{xt}(s)\Big|^m ds\Big\} \leq K(1+|x|^m), \quad \forall x \in \mathbb{R}^n,\ t \in [0,T],\ \forall v \in V_{xt}. \tag{3.22}$$

□

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Lemma 3.2. *If the conditions in* (3.8) *and* (3.13) *are satisfied, then*

$$\begin{cases} (i) \quad k|x^{+}|^{m} - K \leq u(x,t) \leq K(1+|x|^{m}), \\ (ii) \quad |u(x,t) - u(x',t)| \leq K(1+|x|^{m-1} + |x'|^{m-1})|x-x'|. \end{cases} \tag{3.23}$$

Proof. Proof of item (3.23-i). The upper bound of the item in Lemma 3.2(i) follows directly from Lemma 3.1(i). For the proof of the lower bound of Lemma 3.2(i), since there is $c > 0$ such that $\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big) \geq c, \forall s \in [t,T]$, we have

$$\begin{aligned} J_{xt}(v) &\geq \mathbb{E}\Big\{\int_t^T f(y_{xt}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} \\ &\geq c(k\mathbb{E}\Big\{\int_t^T |y_{xt}(s)^{+}|^{m}ds\Big\} - K), \quad \forall v \in V. \end{aligned} \tag{3.24}$$

Applying Lemma 3.1(iv) we easily obtained the result.

Proof of item (3.23-ii).

For any $x, x' \in \mathbb{R}^n, t \in [0,T]$, we obtain

$$|u(x,t) - u(x',t)| \leq \sup\Big\{\Big|J_{xt}(v) - J_{x't}(v)\Big| : \forall v \in V_{xt} \cup V_{x't}\Big\}. \tag{3.25}$$

But

$$|J_{xt}(v) - J_{x't}(v)| \leq \mathbb{E}\Big\{\int_t^T |f(y_{xt}(s),s) - f(y_{x't}(s),s)|\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\}.$$

Using the assumption (3.13-ii), we have

$$|J_{xt}(v) - J_{x't}(v)| \leq K\mathbb{E}\Big\{\int_t^T (1+|y_{xt}(s)|^{m-1} + |y_{x't}(s)|^{m-1})|y_{xt}(s) - y_{x't}(s)|ds\Big\}.$$

From (3.2), we obtain $|y_{xt}(s) - y_{x't}(s)| = |x - x'|$, hence

$$|J_{xt}(v) - J_{x't}(v)| \leq K\mathbb{E}\Big\{\int_t^T (1+|y_{xt}(s)|^{m-1} + |y_{x't}(s)|^{m-1})|x-x'|ds\Big\}. \tag{3.26}$$

From (3.22), we also have

$$\mathbb{E}\Big\{\int_t^T \big|y_{xt}(s)\big|^{m}ds\Big\} \leq K(1+|x|^{m} + |x'|^{m}), \quad \forall x, x' \in \mathbb{R}^n,\ t \in [0,T],\ \forall v \in V_{xt} \cup V_{x't}.$$

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From this estimate, together with Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\Big\{\int_t^T |y_{xt}(s)|^{m-1}ds\Big\} \leq& \quad \Big(\mathbb{E}\Big\{\int_t^T |y_{xt}(s)|^{m}ds\Big\}\Big)^{\frac{m-1}{m}}\Big(\mathbb{E}\Big\{\int_t^T 1ds\Big\}\Big)^{\frac{1}{m}} \\ \leq& \qquad C\Big(K(1+|x|^m+|x'|^m)\Big)^{\frac{m-1}{m}} \\ \leq& \qquad K(1+|x|^{m-1}+|x'|^{m-1}).\end{aligned} \tag{3.27}$$

Since x, x' have the same role, we also have

$$\mathbb{E}\Big\{\int_t^T |y_{x't}(s)|^{m-1}ds\Big\} \leq K(1+|x|^{m-1}+|x'|^{m-1}). \tag{3.28}$$

Substitute (3.27), (3.28) into (3.26) , then we proved Lemma 3.2(ii). □

Lemma 3.3. *Under the conditions* (3.8) *and* (3.13), $u(x,t)$ *is convex in* x *for every fixed* $t \in [0,T]$ *and* $0 \leq \dfrac{\partial^2 u}{\partial x_i^2}(x,t) \leq K(1+|x|^q)$ *for* $q = (m-2)^+, \quad i = 1, \cdots, n.$

Proof.

Part 1. Prove that u is convex. To show u is convex means we have to show $u(\theta x + (1-\theta)x', t) \leq \theta u(x,t) + (1-\theta)u(x',t)$. However, because of the formula of u in (3.4), it suffices to prove

$$J_{\theta x+(1-\theta)x',t}(\theta v + (1-\theta)v') \leq \theta J_{xt}(v) + (1-\theta)J_{x't}(v'), \tag{3.29}$$

for every $x,\ x' \in \mathbb{R}^n,\ t \in [0,T],\ v,\ v' \in V$ and $0 \leq \theta \leq 1$. Since

$$y_{xt}(s,v) = x + v(s-t) + \int_t^s g(\lambda)d\lambda + \int_t^s \sigma(\lambda)dw(\lambda - t)$$

and

$$y_{x't}(s,v') = x + v'(s-t) + \int_t^s g(\lambda)d\lambda + \int_t^s \sigma(\lambda)dw(\lambda - t).$$

Put $x'' = \theta x + (1-\theta)x'$ then

$$y_{x''t}(s, \theta v + (1-\theta)v') = \theta y_{xt}(s,v) + (1-\theta)y_{x't}(s,v').$$

Since f is convex then

$$\begin{aligned} f(y_{x''t}(s,\theta v + (1-\theta)v'), s) &= f(\theta y_{xt}(s,v) + (1-\theta)y_{x't}(s,v'), s) \\ &\leq \theta f(y_{xt}(s,v), s) + (1-\theta)f(y_{x't}(s,v'), s). \end{aligned} \tag{3.30}$$

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From the inequality (3.30), (3.29) is proved. Since $u(\theta x+(1-\theta)x', t) \leq J_{\theta x+(1-\theta)x',t}(\theta v+(1-\theta)v')$ we have for any v and v' that

$$u(\theta x+(1-\theta)x', t) \leq \theta J_{xt}(v)+(1-\theta)J_{x't}(v'). \tag{3.31}$$

Taking the infimum over $v, v' \in V$ we obtain the convexity of u.

Part 2. We prove the existence of the generalized derivatives $\dfrac{\partial^2 u}{\partial x_i^2}(x,t)$ and $\dfrac{\partial u}{\partial t}(x,t)$ as well as the estimate

$$0 \leq \frac{\partial^2 u}{\partial x_i^2}(x,t) \leq K(1+|x|^q), \quad q=(m-2)^+, \quad i=1,\cdots,n. \tag{3.32}$$

Let h_i be a vector having the i-th component $h \in \mathbb{R}$ and the other components 0. We will show

$$\begin{aligned}&u(x+h_i,t)-2u(x,t)+u(x-h_i,t)\\ &\leq \sup\{J_{x+h_i,t}(v)-2J_{x,t}(v)+J_{x-h_i,t}(v) : v \in V \text{ satisfying } (3.40)\}.\end{aligned} \tag{3.33}$$

Since

$$u(x+h_i,t)-2u(x,t)+u(x-h_i,t) = u(x+h_i,t)-u(x,t)+u(x-h_i,t)-u(x,t) \tag{3.34}$$

then it suffices to prove

$$u(x+h_i,t)-u(x,t) \leq \sup\{J_{x+h_i,t}(v)-J_{x,t}(v)\}. \tag{3.35}$$

By the definition of infimum, for all ϵ there exist v^ϵ such that

$$u(x,t) \leq J_{x,t}(v^\epsilon) \leq u(x,t)+\epsilon,$$

then

$$-u(x,t) \leq -J_{x,t}(v^\epsilon)+\epsilon.$$

Therefore

$$\begin{aligned}u(x+h_i,t)-u(x,t) \leq\ & J_{x+h_i,t}(v^\epsilon)-J_{x,t}(v^\epsilon)+\epsilon\\ \leq\ & \sup\{J_{x+h_i,t}(v)-J_{x,t}(v)\}+\epsilon.\end{aligned} \tag{3.36}$$

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Let $\epsilon \to 0_+$ then we get (3.35). We have

$$f(z+h_i,s)-2f(z,s)+f(z-h_i,s)=h^{-2}\int_0^1 d\lambda \int_{-\lambda}^{\lambda} \frac{\partial^2 f}{\partial x_i^2}(z+\mu h_i,s)d\mu, \tag{3.37}$$

and

$$y_{x\pm h_i,t}(s)=y_{xt}(s)\pm h_i. \tag{3.38}$$

By Lemma 3.1, we restrict admissible controls to those satisfying $\mathbb{E}\Big\{|y_{xt}(s)|^m\Big\} \leq K(1+|x|^m)$. Applying Hölder's inequality we obtain $\mathbb{E}\Big\{|y_{xt}(s)|^q\Big\} \leq K(1+|x|^q)$. Using the hypothesis $0 \leq \dfrac{\partial^2 f}{\partial x_i^2}(x,s) \leq K(1+|x|^q)$ and (3.37), (3.38) we have

$$\begin{aligned} &J_{x+h_i,t}(v)-2J_{x,t}(v)+J_{x-h_i,t}(v) \\ &\quad = \mathbb{E}\Big\{\int_t^T \Big[f(y_{(x+h_i)t}(t),t)-2f(y_{xt}(s),s)+f(y_{(x-h_i)t}(s)\Big]\exp(-\int_t^s \alpha(\lambda)d\lambda)\Big\} \\ &\quad \leq K(1+|x|^q). \end{aligned}$$

As a result,

$$u(x+h_i,t)-2u(x,t)+u(x-h_i,t)\leq K(1+|x|^q)|h_i|^2. \tag{3.39}$$

Let B be any open ball in $\mathbb{R}^n$ and let $\phi \in C_0^\infty(\mathbb{R}^n\times\mathbb{R})$ be any test function with compact support. Let h_i be a vector having the i-th component $h\in\mathbb{R}$ and the other components 0. Since $u(x+h_i,t)-2u(x,t)+u(x-h_i,t)\leq K(1+|x|^q)h^2$ for $|h|\leq 1$ (by (3.39)), there is a sequence $h^{(k)}\to 0_+$ as $k\to\infty$ such that, denoting $g_k=(h^{(k)})^{-2}\big[u(x+h_i^{(k)},t)-2u(x,t)+u(x-h_i^{(k)},t)\big]$, we have $g_k\to Q$ weakly in $L^p(B\times[0,T])$ for some p with $1<p<\infty$. It is easy to show that

$$\int_0^T\int_{\mathbb{R}^n}\phi(x,t)Q(x,t)dxdt=\int_0^T\int_{\mathbb{R}^n}\frac{\partial^2\phi}{\partial x_i^2}u(x,t)dxdt, \quad \forall\phi\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}),\ \mathrm{supp}(\phi)\text{ in } B\times[0,T],$$

where $Q=\dfrac{\partial^2 u}{\partial x_i^2}$ is a generalized derivative. As a result, Q is the generalized derivative $\dfrac{\partial^2 u}{\partial x_i^2}$. Taking the limit of (3.39) we deduce that $\dfrac{\partial^2 u}{\partial x_i^2} \leq K(1+|x|^q)$ then (3.23-iii) is proved. The existence and local boundedness of mixed second order generalized derivatives can be proved as follow. For

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$k = 1, \cdots, n$, let e_k denote the unit vector in the direction of the positive x_k axis. For any fixed $i \neq j$ with $1 \leq i, j \leq n$, let y be a new coordinate whose axis points in the $\dfrac{(e_i + e_j)}{\sqrt{2}}$ direction. Then $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \dfrac{\partial^2 u}{\partial y^2} - \Big(\dfrac{\partial^2 u}{\partial x_i^2} + \dfrac{\partial^2 u}{\partial x_j^2}\Big)\Big/2.$ □

Lemma 3.4. *Suppose that* (3.8) *and* (3.13) *hold. If the optimal control* $u(x,t)$ *satisfies*

$$\begin{aligned} u(x,t) \leq \mathbb{E}\Big\{ \int_t^{t'} f(y_{xt}^0(s), s) \exp\Big(- \int_t^s \alpha(\lambda) d\lambda \Big) ds \\ + u(y_{xt}^0(t'), t') \exp\Big(- \int_t^{t'} \alpha(s) ds \Big) \Big\}, \end{aligned} \tag{3.40}$$

where $y_{xt}^0(s)$ *is given by* (3.2) *with* $v = 0$, *then*

$$|u(x,t) - u(x,t')| \leq K(1 + |x|^m)|t - t'| \tag{3.41}$$

for every $(x,t), (x',t') \in \mathbb{R}^n \times [0,T]$ *and some constant* K. *As a result,* $\dfrac{\partial u}{\partial t}(x,t) \leq K(1 + |x|^m)$. *Moreover, we have*

$$\frac{\partial u}{\partial t} \in L^\infty_{loc}(\mathbb{R}^n \times [0,T]). \tag{3.42}$$

Proof.

Proof of (3.41). We observe that

$$\begin{aligned} J_{xt}(v) = \mathbb{E}\Big\{ \int_0^{T-t} f(y_{xt}(t+s), t+s) \exp\Big(- \int_0^s \alpha(t+\lambda) d\lambda \Big) ds \\ + \sum_{i=1}^n c_i \int_0^{T-t} \exp\Big(- \int_0^s \alpha(t+\lambda) d\lambda \Big) dv_i(s) \Big\} \end{aligned} \tag{3.43}$$

and

$$y_{xt}(s+t) = x + v(s) + \int_0^s g(\lambda + t) d\lambda + \int_0^s \sigma(\lambda + t) dw(\lambda).$$

Similarly, we also have

$$y_{xt'}(s+t') = x + v(s) + \int_0^s g(\lambda + t') d\lambda + \int_0^s \sigma(\lambda + t') dw(\lambda).$$

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Therefore

$$|y_{xt}(s+t) - y_{xt'}(s+t')| \leq \int_0^s |g(\lambda+t) - g(\lambda+t')|d\lambda + \int_0^s |\sigma(\lambda+t) - \sigma(\lambda+t')|dw(\lambda).$$

By the assumptions (3.8) of g and σ we have

$$\mathbb{E}\Big\{|y_{xt}(t+s) - y_{xt'}(t'+s)|^m\Big\} \leq K|t-t'|^m, \tag{3.44}$$

for every s in $[0, T-t]$, and a constant K independent of x, t, t', v. Based on the equation (3.2) and assumptions (3.8), it follows from (3.22) that

$$\mathbb{E}\Big\{\int_0^{T-t} |v(s)|^m ds\Big\} \leq K(1+|x|^m),\ \forall (x,t) \in \mathbb{R}^n \times [0,T]. \tag{3.45}$$

We will consider two cases:

Case 1 : $t' \leq t$. Note that if $\mathbb{E}\Big\{\int_0^{T-t'} |v(s)|^m ds\Big\} \leq K(1+|x|^m)$ then $\mathbb{E}\Big\{\int_0^{T-t} |v(s)|^m ds\Big\} \leq K(1+|x|^m)$. Hence,

$$u(x,t) - u(x,t') \leq \sup\Big\{J_{xt}(v) - J_{xt'}(v) : v \in V \text{ satisfying (3.45)}\Big\}. \tag{3.46}$$

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Moreover,

$$\begin{aligned}
J_{xt}(v) - J_{xt'}(v) = \mathbb{E}\Big\{ \int_0^{T-t} f(y_{xt}(t+s), t+s) \exp\Big(-\int_0^s \alpha(t+\lambda)d\lambda\Big)ds \\
+ \sum_{i=1}^n c_i \int_0^{T-t} \exp\Big(-\int_0^s \alpha(t+\lambda)d\lambda\Big)dv_i(s)\Big\} \\
- \mathbb{E}\Big\{ \int_0^{T-t'} f(y_{xt'}(t'+s), t'+s) \exp\Big(-\int_0^s \alpha(t'+\lambda)d\lambda\Big)ds \\
+ \sum_{i=1}^n c_i \int_0^{T-t'} \exp\Big(-\int_0^s \alpha(t'+\lambda)d\lambda\Big)dv_i(s)\Big\} \\
= \mathbb{E}\Big\{ \int_0^{T-t} \Big[f(y_{xt}(t+s), t+s) \exp\Big(-\int_0^s \alpha(t+\lambda)d\lambda\Big) \qquad (3.47) \\
- f(y_{xt'}(t'+s), t'+s) \exp\Big(-\int_0^s \alpha(t'+\lambda)d\lambda\Big)\Big]ds \\
+ \sum_{i=1}^n c_i \int_0^{T-t} \Big[\exp\Big(-\int_0^s \alpha(t+\lambda)d\lambda\Big) - \exp\Big(-\int_0^s \alpha(t'+\lambda)d\lambda\Big)\Big]dv_i(s)\Big\} \\
- \mathbb{E}\Big\{ \int_{T-t}^{T-t'} f(y_{xt'}(t'+s), t'+s) \exp\Big(-\int_0^s \alpha(t'+\lambda)d\lambda\Big)ds \\
+ \sum_{i=1}^n c_i \int_{T-t}^{T-t'} \exp\Big(-\int_0^s \alpha(t'+\lambda)d\lambda\Big)dv_i(s)\Big\}.
\end{aligned}$$

For $b,\ b' > 0$, we have $ae^{-b} - a'e^{-b'} < |a||b-b'| + |a-a'|$. Therefore

$$\begin{aligned}
\mathbb{E}\Big\{ \int_0^{T-t} \Big[f(y_{xt}(t+s), t+s) \exp\Big(-\int_0^s \alpha(t+\lambda)d\lambda\Big) \\
- f(y_{xt'}(t'+s), t'+s) \exp\Big(-\int_0^s \alpha(t'+\lambda)d\lambda\Big)\Big]ds\Big\} \\
\le \mathbb{E}\Big\{ \int_0^{T-t} \Big|f(y_{xt}(t+s), t+s) - f(y_{xt'}(t'+s), t'+s)\Big|ds\Big\} \qquad (3.48) \\
+ \mathbb{E}\Big\{ \int_0^{T-t} \Big|f(y_{xt}(t+s), t+s)\Big| \int_0^s \Big|\alpha(t+\lambda) - \alpha(t'+\lambda)\Big|d\lambda ds\Big\}.
\end{aligned}$$

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Applying (3.13-i) and (3.13-ii) to (3.48) we have

$$
\begin{aligned}
&\mathbb{E}\Big\{ \int_0^{T-t} \Big[f(y_{xt}(t+s), t+s) \exp\Big(- \int_0^s \alpha(t+\lambda)d\lambda \Big) \\
&\qquad - f(y_{xt'}(t'+s), t'+s) \exp\Big(- \int_0^s \alpha(t'+\lambda)d\lambda \Big)\Big] ds \Big\} \\
&\leq K\mathbb{E}\Big\{ \int_0^{T-t} \Big(1 + |y_{xt}(t+s)|^{m-1} + |y_{xt'}(t+s)|^{m-1}\Big)|y_{xt}(t+s) - y_{xt'}(t'+s)|ds \Big\} \\
&\quad + K\mathbb{E}\Big\{ \int_0^{T-t} \Big(1 + |y_{xt}(t+s)|^m\Big) ds \Big\} \cdot |t-t'|.
\end{aligned}
\tag{3.49}
$$

On the other hand, by (3.13-i),

$$
\begin{aligned}
&\mathbb{E}\Big\{ \int_{T-t}^{T-t'} f(y_{xt'}(t'+s), t'+s) \exp\Big(- \int_0^s \alpha(t'+\lambda)d\lambda \Big) ds \\
&\quad + \sum_{i=1}^n c_i \int_{T-t}^{T-t'} \exp\Big(- \int_0^s \alpha(t'+\lambda)d\lambda \Big) dv_i(s) \Big\} \\
&\geq \mathbb{E}\Big\{ \int_{T-t}^{T-t'} \big(k|y^+_{xt'}(t'+s)|^m - K\big) \exp\Big(- \int_0^s \alpha(t'+\lambda)d\lambda \Big) ds \Big\} \geq -K|t-t'|.
\end{aligned}
\tag{3.50}
$$

We also have

$$
\begin{aligned}
&\mathbb{E}\Big\{ \sum_{i=1}^n c_i \int_0^{T-t} \Big[\exp\Big(- \int_0^s \alpha(t+\lambda)d\lambda \Big) - \exp\Big(- \int_0^s \alpha(t'+\lambda)d\lambda \Big)\Big] dv_i(s) \Big\} \\
&\qquad \leq \mathbb{E}\Big\{ \sum_{i=1}^n c_i \int_0^{T-t} K|t-t'|dv_i(s) \Big\} = K\mathbb{E}\{|v(T-t)|\}|t-t'|.
\end{aligned}
\tag{3.51}
$$

From (3.49), (3.50) and (3.51), we have

$$
\begin{aligned}
J_{xt}(v) - J_{xt'}(v) \leq{} & K\mathbb{E}\Big\{ \Big[\int_t^T (1 + |y_{xt}(s)|^m) ds + v(T-t) \Big] |t-t'| \\
& + \int_0^{T-t} (1 + |y_{xt}(t+s)|^{m-1} + |y_{xt'}(t'+s)|^{m-1})|y_{xt}(t+s) - y_{xt'}(t'+s)|ds \Big\}.
\end{aligned}
\tag{3.52}
$$

The first part of (3.52) can be estimated by using (3.18) and (3.22)

$$
\mathbb{E}\Big\{ \Big[\int_t^T (1 + |y_{xt}(s)|^m) ds + v(T-t) \Big] |t-t'| \Big\} \leq K(1 + |x|^m)|t-t'|. \tag{3.53}
$$

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Applying Hölder's inequality and using (3.44), the second part of (3.52) can be reduced to verifying the following inequality

$$\begin{aligned} &\mathbb{E}\Big\{\int_0^{T-t} |y_{xt}(t+s)|^{m-1}|y_{xt}(t+s)-y_{xt'}(t'+s)|ds\Big\} \\ &\le \Big(\mathbb{E}\Big\{\int_0^{T-t} |y_{xt}(t+s)|^{m-1}ds\Big\}\Big)^{\frac{m}{m-1}}\Big(\mathbb{E}\Big\{\int_0^{T-t} |y_{xt}(t+s)-y_{xt'}(t'+s)|^m ds\Big\}\Big)^{\frac{1}{m}} \\ &\le K(1+|x|^m)|t-t'|. \end{aligned} \tag{3.54}$$

In summary, we have

$$u(x,t)-u(x,t') \le K(1+|x|^m)|t-t'|, \quad t' \le t. \tag{3.55}$$

Case 2 : $t' > t$.

Since $y^0(s)$ corresponds to the free evolution, $v = 0$, then we have for $K > 0$,

$$\mathbb{E}\Big\{|y^0_{xt}(s)|^m\Big\} \le K(1+|x|^m), \ \forall s \in [t,t']. \tag{3.56}$$

Fix $t' > t$, then apply the Ito's formula for a function $\phi(x)$ satisfying

$$\Big|\frac{\partial \phi}{\partial x}\Big| \le K(1+|x|^{m-1}) \tag{3.57}$$

and

$$\Big|\frac{\partial^2 \phi}{\partial x^2}\Big| \le K(1+|x|^q), \quad q=(m-2)^+, \tag{3.58}$$

we have

$$\begin{aligned} d\phi(y^0_{xt}(s)) = \Big(\frac{\partial \phi}{\partial x}\cdot g(s) + \frac{1}{2}\mathrm{tr}\frac{\partial^2\phi}{\partial x^2}\cdot \sigma^T\sigma\Big)ds \\ + \frac{\partial \phi}{\partial x}\cdot \sigma dw(s-t). \end{aligned} \tag{3.59}$$

Thus

$$\mathbb{E}\Big\{\phi(y^0_{xt}(s))\Big\} = \phi(x) + \mathbb{E}\Big\{\int_t^{t'} \Big(\frac{\partial \phi}{\partial x}\cdot g(s) + \frac{1}{2}\mathrm{tr}\frac{\partial^2\phi}{\partial x^2}\cdot \sigma^T\sigma\Big)ds\Big\}. \tag{3.60}$$

It follows from (3.57) and (3.58) that

$$\mathbb{E}\Big\{\phi(y^0_{xt}(s))\Big\} \le \phi(x) + K\mathbb{E}\Big\{\int_t^{t'} \Big(1+|y^0_{xt}(s)|^m\Big)ds\Big\}. \tag{3.61}$$

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Due to (3.23) we can take a sequence of functions $\phi^n(x)$ twice continuously differentiable that converges to $u(x,t')$. From (3.61), we have

$$\mathbb{E}\Big\{u(y^0_{xt}(s),t')\Big\} = u(x,t') + K\mathbb{E}\Big\{\int_t^{t'}\Big(1+|y^0_{xt}(s)|^m\Big)ds\Big\}. \tag{3.62}$$

Therefore,

$$\begin{aligned} u(x,t)-u(x,t') \leq\ & \mathbb{E}\Big\{\int_t^{t'} f(y^0_{xt}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds \\ & + u(y^0_{xt}(t'),t')\exp\Big(-\int_t^{t'}\alpha(s)ds\Big)\Big\} - u(x,t') \\ & \leq \mathbb{E}\Big\{\int_t^{t'} f(y^0_{xt}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} \\ & + \Big(K\mathbb{E}\Big\{\int_t^{t'}\Big(1+|y^0_{xt}(s)|^m\Big)ds\Big\} + u(x,t')\Big)\exp\Big(-\int_t^{t'}\alpha(s)ds\Big) \\ & - u(x,t') \\ & = \mathbb{E}\Big\{\int_t^{t'} f(y^0_{xt}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} \\ & + K\mathbb{E}\Big\{\int_t^{t'}\Big(1+|y^0_{xt}(s)|^m\Big)ds\Big\}\exp\Big(-\int_t^{t'}\alpha(s)ds\Big) \\ & + u(x,t')\Big(\exp\Big(-\int_t^{t'}\alpha(s)ds\Big)-1\Big) \\ & = I_1 + I_2 + I_3. \end{aligned} \tag{3.63}$$

The term I_1 can be verified by the following

$$I_1 := \mathbb{E}\Big\{\int_t^{t'} f(y^0_{xt}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\} \leq K(1+|x|^m)|t-t'|. \tag{3.64}$$

Similarly the term I_2 can be verified by the following

$$I_2 := K\mathbb{E}\Big\{\int_t^{t'}\Big(1+|y^0_{xt}(s)|^m\Big)ds\Big\}\exp\Big(-\int_t^{t'}\alpha(s)ds\Big) \leq K(1+|x|^m)|t-t'|. \tag{3.65}$$

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To verify the term I_3, first we have to use the mean value theorem, let $g(t') = \exp\Big(-\int_t^{t'} \alpha(s)ds\Big)$, we have

$$g(t') - g(t) = \Big(\exp\Big(-\int_t^{t'} \alpha(s)ds\Big) - 1\Big) \leq K(t-t').$$

Using (3.57), the term I_3 can be verified by the following

$$I_3 := u(x,t')\Big(\exp\Big(-\int_t^{t'} \alpha(s)ds\Big) - 1\Big) \leq K(1+|x|^m)|t-t'|. \tag{3.66}$$

Hence,

$$|u(x,t) - u(x,t')| \leq K(1+|x|^m)|t-t'|, \quad t' > t. \tag{3.67}$$

Combining both **Case 1** and **Case 2**, then (3.41) is proved.

Finally, we will prove that

$$\frac{\partial u}{\partial t} \in L^\infty_{loc}(\mathbb{R}^n \times [0,T]). \tag{3.68}$$

From $|u(x,t) - u(x,t+\Delta t)| \leq K(1+|x|^m)\Delta t$ (by (3.41)), by the same limitation arguments as in Lemma 3.3, we have

$$\int_0^T \int_{\mathbb{R}^n} \phi(x,t)P(x,t)dxdt = -\int_0^T \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial t} u(x,t)dxdt, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}),\ \text{supp}(\phi) \text{ in } B \times [0,T],$$

where $P = \dfrac{\partial u}{\partial t}$. □

Taking into account the above lemmas, we have the following theorem:

Theorem 3.1. *Under the assumptions* (3.8), (3.13) *the optimal cost* u *defined by* (3.4) *is a non-negative continuous function such that*

$$\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^\infty_{loc}(\mathbb{R}^n \times [0,T]), \quad i,j = 1,\cdots,n, \tag{3.69}$$

and for some other constants $0 < k \leq K$,

$$\begin{cases} (i) \quad k|x^+|^m - K \leq u(x,t) \leq K(1+|x|^m), \\ (ii) \quad |u(x,t) - u(x',t)| \leq K(1+|x|^{m-1} + |x'|^{m-1})|x-x'|, \\ (iii) \quad u \text{ is convex in } x \text{ for every fixed } t \text{ in } [0,T], \text{ with} \\ \qquad 0 \leq \dfrac{\partial^2 u}{\partial x_i^2}(x,t) \leq K(1+|x|^q), \quad q = (m-2)^+, \quad i = 1,\cdots,n. \end{cases} \tag{3.70}$$

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Moreover, if u satisfies

$$u(x,t) \leq \mathbb{E}\Big\{ \int_t^{t'} f(y^0_{xt}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds + u(y^0_{xt}(t'),t')\exp\Big(-\int_t^{t'} \alpha(s)ds\Big)\Big\}, \tag{3.71}$$

where $y^0_{xt}(s)$ is given by (3.2) with $v=0$ then

$$|u(x,t)-u(x,t')| \leq K(1+|x|^m)|t-t'| \tag{3.72}$$

for every $(x,t),(x',t') \in \mathbb{R}^n \times [0,T]$ and some constant K.

3.4 Penalized Equation

It is difficult to investigate directly the solution of (5.7). Thus, it is natural to consider the penalized equation, in which the coefficients are smooth.

$$\begin{cases} Au^\epsilon + \dfrac{1}{\epsilon}\displaystyle\sum_{i=1}^n \beta\Big(\frac{\partial u^\epsilon}{\partial x_i}+c_i\Big) = f, \ x \in \mathbb{R}^n, \ t \in [0,T], \\ u^\epsilon(x,T) = 0, \end{cases} \tag{3.73}$$

with $\beta \in C^\infty(\mathbb{R})$, β convex, nonincreasing, and satisfy

$$\beta(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda \geq 0, \\ -2\lambda - 1 & \text{if} \quad \lambda \leq -1, \\ \text{positive} & \text{if} \quad \lambda < 0. \end{cases} \tag{3.74}$$

Let V_ϵ, $\epsilon > 0$ denote the set of all progressively measurable random processes $\eta(t)$, $\xi(t)$ from $[0,\infty[$ into $\mathbb{R}^n$ whose components $\eta_i(t)$, $\xi_i(t)$ are nonnegative and satisfy for $1 \leq i \leq n$, $t \geq 0$, $s \in \mathbb{R}$,

$$-s\eta_i(t) - \frac{1}{\epsilon}\beta(s) \leq \xi_i(t) \leq \frac{1}{\epsilon}.$$

Note that (η,ξ) belongs to V_ϵ, then for $t \geq 0$,

$$0 \leq \eta_i(t) \leq \frac{2}{\epsilon}, \quad 0 \leq \xi_i(t) \leq \frac{1}{\epsilon}. \tag{3.75}$$

Let

$$J_{xt}(\eta,\xi) = J_{xt}(v) \ + \ \mathbb{E}\Big\{\int_t^T \sum_{i=1}^n \xi_i(s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big\}, \tag{3.76}$$

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with

$$v_i(s) = \int_0^t \eta_i(s)ds. \tag{3.77}$$

3.5 Some results of the approximating functions u_ϵ

Define

$$u^\epsilon(x,t) = \inf\{J_{xt}(\eta,\xi) : (\eta,\xi) \in V_\epsilon\}. \tag{3.78}$$

Lemma 3.5. *If f is continuously differentiable and g is increasing on the interval $[a,b]$, then we have the integration by parts*

$$\int_a^b f(t)dg(t) = f(b)g(b) - f(a)g(a) - \int_a^b g(t)f'(t)dt.$$

Proof. Consider a partition P: $t_0 = a < t_1 < \cdots < t_n = b$. Let $\delta(P) = \max_{i=1,\ldots,n}\{t_i - t_{i-1}\}$, we have: $\int_a^b f(t)dg(t) = \lim_{\delta(P)\longrightarrow 0} \sum_{i=1}^n f(\xi_{i-1})[g(t_i) - g(t_{i-1})]$ for any $\xi_i \in [t_{i-1}, t_i]$. We can write:

$$\begin{aligned} S(P) &= \sum_{i=1}^n f(\xi_{i-1})[g(t_i) - g(t_{i-1})] \\ &= \sum_{i=1}^n f(t_{i-1})[g(t_i) - g(t_{i-1})] + \sum_{i=1}^n [f(\xi_{i-1}) - f(t_{i-1})][g(t_i) - g(t_{i-1})]. \end{aligned}$$

Since f is uniformly continuous, we have

$$\begin{aligned} &\Big|\sum_{i=1}^n [f(\xi_{i-1}) - f(t_{i-1})][g(t_i) - g(t_{i-1})]\Big| \\ &\quad \le \sup_{i=1,\ldots,n}\{|f(\xi_{i-1}) - f(t_{i-1})|\}\sum_{i=1}^n [g(t_i) - g(t_{i-1})] \\ &\quad = \sup_{i=1,\ldots,n}\{|f(\xi_{i-1}) - f(t_{i-1})|\} \times (g(b) - g(a)) \longrightarrow 0 \text{ as } \delta(P) \longrightarrow 0. \end{aligned} \tag{3.79}$$

On the other hand,

$$\begin{aligned} \sum_{i=1}^n f(t_{i-1})[g(t_i) - g(t_{i-1})] &= f(t_{n-1})g(t_n) - f(t_0)g(t_0) - \sum_{i=1}^n [f(t_i) - f(t_{i-1})]g(t_i) \\ &\longrightarrow f(b)g(b) - f(a)g(a) - \int_a^b g(t)df(t) \text{ as } \delta(P) \longrightarrow 0. \end{aligned} \tag{3.80}$$

The two limitations (3.79), (3.80) give us the desired results. □

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Lemma 3.6. *If* (3.8), (3.13) *are satisfied, there exist positive constants* $0 < k \leq K,\ m \geq 1$, *the optimal cost* u^ϵ *given by* (3.78) *satisfies the following:*

$$\begin{cases} (i) \quad k|x^+|^m - K \leq u^\epsilon(x,t) \leq K(1+|x|^m), \\ (ii) \quad |u^\epsilon(x,t) - u^\epsilon(x',t)| \leq K(1+|x|^{m-1}+|x'|^{m-1})|x-x'|, \\ (iii) \quad |u^\epsilon(x,t) - u^\epsilon(x,t')| \leq K(1+|x|^m)|t-t'|, \\ \dfrac{\partial u^\epsilon}{\partial t}, \dfrac{\partial^2 u^\epsilon}{\partial x_i \partial x_j} \in L^\infty_{loc}(\mathbb{R}^n \times [0,T]),\ u^\epsilon \text{ is convex, and} \\ 0 \leq \dfrac{\partial^2 u^\epsilon}{\partial x_i^2}(x,t) \leq K(1+|x|^q),\ \text{with } q = (m-2)^+,\ i = 1, \cdots, n \\ \text{for every } (x,t),(x',t') \in \mathbb{R}^n \times [0,T]. \end{cases} \tag{3.81}$$

Proof. Step 1. Proof of item (3.81-i).

Consider $\eta = 0, \xi = 0$ then $u^\epsilon \leq J_{xt}(0,0) \leq K(1+|x|^m)$. Obviously,

$$J_{xt}(\eta,\xi) \geq J_{xt}(v)$$

where $v(t) = \int_0^t \eta(s)ds$. By Theorem 3.1 (i), we have $J_{xt}(v) \geq u(x,t) \geq k|x^+|^m - K$ for any v. As a result, $J_{xt}(\eta,\xi) \geq k|x^+|^m - K$ for any $(\eta,\xi) \in V_\epsilon$. The item (i) is therefore proved.

Step 2. Proof of item (3.81-ii).

Let

$$V^*_{xt} = \Big\{(\eta,\xi) \in V_\epsilon : \mathbb{E}\Big\{\int_0^{T-t} |v(s)|^m ds\Big\} \leq K(1+|x|^m),\ \forall (x,t) \in \mathbb{R}^n \times [0,T]\Big\}. \tag{3.82}$$

Similar to Theorem 3.1, we have $u^\varepsilon(x,t) = \inf_{(\eta,\xi)\in V^*_{xt}} \{J_{xt}(\eta,\xi)\}$. Hence,

$$|u^\epsilon(x,t) - u^\epsilon(x',t)| \leq \sup\Big\{\Big|J_{xt}(\eta,\xi) - J_{x't}(\eta,\xi)\Big| : \forall v \in V^*_{xt} \cup V^*_{x't}\Big\}. \tag{3.83}$$

Let

$$S := \sup\Big\{\Big|J_{xt}(\eta,\xi) - J_{x't}(\eta,\xi)\Big| : \forall v \in V^*_{xt} \cup V^*_{x't}\Big\}.$$

Thus

$$J_{xt}(\eta,\xi) - J_{x't}(\eta,\xi) \leq S,$$

implies

$$\inf_{V^*_{xt}\cup V^*_{x't}} J_{x't}(\eta,\xi) \leq \inf_{V^*_{xt}\cup V^*_{x't}} J_{xt}(\eta,\xi) + S.$$

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Therefore

$$u^{\epsilon}(x,t) - u^{\epsilon}(x',t) \leq S. \tag{3.84}$$

Following the same proof of of item (3.23-ii), we can prove (3.81-ii).

Step 3. Proof of item (3.81-iii).

Case 1 : $t' \leq t$. We begin with a remark that if $-s\eta_i(t) - \frac{1}{\epsilon}\beta(s) \leq \xi_i(t) \leq \frac{1}{\epsilon},\ \forall s \in \mathbb{R}$, then we also have

$$-s\eta_i(t) - \frac{1}{\epsilon}\beta(s) \leq \tilde{\xi}_i(t) \leq \frac{1}{\epsilon},\ \forall s \in \mathbb{R},$$

where $\tilde{\xi}_i(t) = \min\Big\{\xi_i(t),\ \eta_i(t),\ \frac{1}{\epsilon}\Big\}$. Moreover, since $\tilde{\xi}_i(t) \leq \xi_i(t)$, we have $J_{xt}(\eta,\tilde{\xi}) \leq J_{xt}(\eta,\xi)$. On the other hand, since $f(x,t) \geq -K,\ \forall (x,t) \in \mathbb{R}^n \times [0,T]$ and $\alpha(t)$ is strictly positive, we have

$$J_{xt}(\eta,\xi) \geq \sum_{i=1}^{n} c_i \int_0^{T-t} \exp\Big(-\int_0^{s} \alpha(t+\lambda)d\lambda\Big)dv_i(s) \geq c\sum_{i=1}^{n} v_i(T-t)$$

for some $c > 0$. Moreover, the value of $v_i(s)$ in the interval $[T-t,T]$ does not matter to the value of J_{xt} (we can take $v_i(s) = v_i(T-t), \forall\, s \geq T-t$). For this reason we can restrict the set of admissible controls to those satisfying $\mathbb{E}\{\sum_{i=1}^{n} v_i(T)\} \leq K(1+|x|^m)$. In summary, we have we have $u^{\epsilon}(x,t) = \inf\{J_{xt}(\eta,\xi) : (\eta,\xi) \in V^{*}_{\varepsilon xt}\}$ where

$$\begin{aligned} V^{*}_{\varepsilon xt} = \Big\{(\eta,\xi) \in V_{\epsilon} : \mathbb{E}\Big\{\int_0^{T-t} |v(s)|^m ds)\Big\}, \leq K(1+|x|^m) \\ \text{for } v_i(s) = \int_0^{t} \eta_i(s)ds,\ \mathbb{E}\{\sum_{i=1}^{n} v_i(T)\} \leq K(1+|x|^m) \\ \text{and } \xi_i(t) \leq \min\Big\{\frac{1}{\epsilon},\eta_i(t)\Big\},\ \forall (x,t) \in \mathbb{R}^n \times [0,T]\Big\}. \end{aligned}$$

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It is obvious that $V^*_{\varepsilon x t'} \subset V^*_{\varepsilon x t}$.

$$\begin{aligned}
&u^\epsilon(x,t) - u^\epsilon(x,t') \\
&\le \sup\Big\{\Big(J_{xt}(v) - J_{xt'}(v)\Big) : v \in V \text{ satisfying } (3.45)\Big\} \\
&\quad + \sup_{(\eta,\xi)\in V^*_{\varepsilon x t}} \mathbb{E}\Big\{\Big|\int_t^T \sum_{i=1}^n \xi_i(s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds - \int_{t'}^T \sum_{i=1}^n \xi_i(s)\Big(\exp\Big(-\int_{t'}^s \alpha(\lambda)d\lambda\Big)ds\Big|\Big\} \\
&= \sup\Big\{\Big(J_{xt}(v) - J_{xt'}(v)\Big) : v \in V \text{ satisfying } (3.45)\Big\} \\
&\quad + \sup_{(\eta,\xi)\in V^*_{\varepsilon x t}} \mathbb{E}\Big\{\Big(-\int_{t'}^t \sum_{i=1}^n \xi_i(s)\exp\Big(-\int_{t'}^s \alpha(\lambda)d\lambda\Big)ds \qquad (3.85)\\
&\quad + \int_t^T \sum_{i=1}^n \xi_i(s)\Big(\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big) - \exp\Big(-\int_{t'}^s \alpha(\lambda)d\lambda\Big)\Big)ds\Big)\Big\} \\
&\le \sup\Big\{\Big(J_{xt}(v) - J_{xt'}(v)\Big) : v \in V \text{ satisfying } (3.45)\Big\} \\
&\quad + \sup_{(\eta,\xi)\in V^*_{\varepsilon x t}} \mathbb{E}\Big\{\Big(\int_t^T \sum_{i=1}^n \xi_i(s)\Big(\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big) - \exp\Big(-\int_{t'}^s \alpha(\lambda)d\lambda\Big)\Big)ds\Big)\Big\}.
\end{aligned}$$

We can estimate

$$\begin{aligned}
\Big|\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big) - \exp\Big(-\int_{t'}^s \alpha(\lambda)d\lambda\Big)\Big| &\le \Big|\int_t^s \alpha(\lambda)d\lambda - \int_{t'}^s \alpha(\lambda)d\lambda\Big| \\
&= \Big|\int_t^{t'} \alpha(\lambda)d\lambda\Big| \le K|t-t'|.
\end{aligned} \qquad (3.86)$$

Hence

$$\begin{aligned}
\mathbb{E}\Big\{\int_t^T \sum_{i=1}^n \xi_i(s)\Big(\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big) - \exp\Big(-\int_{t'}^s \alpha(\lambda)d\lambda\Big)\Big)ds\Big|\Big\}& \\
\le \mathbb{E}\Big\{\int_0^{T-t} \eta(s)ds\Big\}\cdot K|t-t'|& \qquad (3.87)\\
\le K\mathbb{E}\{v(T)\}|t-t'| \le K(1+|x|^m)|t-t'|.&
\end{aligned}$$

By the proof of Theorem 3.1,

$$\sup\Big\{\Big(J_{xt}(v) - J_{xt'}(v)\Big) : v \in V \text{ satisfying } (3.45)\Big\} \le K(1+|x|^m)|t-t'|. \qquad (3.88)$$

Combining (3.88) and (3.87) to apply to (3.85), we have the desired conclusion for the case $t' \le t$.

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Case 2 : $t' > t$. This case can be proved similarly as in the proof of Theorem 3.1. The other claims are proved in the same manner as in Theorem 3.1. □

Lemma 3.7. *If the conditions* (3.8), (3.13) *hold, then for each* $(x,t) \in \mathbb{R}^n \times [0,T]$, *we have* $u^\epsilon(x,t) \longrightarrow u(x,t)$ *as* $\epsilon \longrightarrow 0$.

Proof. Denote by $y(s), y'(s)$ the output corresponding to control v, v' in V, we obtain

$$\int_t^T |y(s)-y'(s)|^m ds = \int_0^{T-t} |v(s)-v'(s)|^m ds. \tag{3.89}$$

Suppose an arbitrary control v in V is given. We define

$$v^{(k)}(t) = \begin{cases} (1-kt)v(0) + k^2 t \displaystyle\int_0^{\frac{1}{k}} v(s)ds & \text{if } 0 \le t \le \dfrac{1}{k}, \\ k \displaystyle\int_{t-\frac{1}{k}}^{t} v(s)ds & \text{otherwise.} \end{cases} \tag{3.90}$$

We first show that $v^{(k)}(t)$ is Lipschitz. Indeed, if $0 \le t \le \dfrac{1}{k}$ then $v^{(k)}(t)$ is linear, then implies Lipschitz. Otherwise, if $\dfrac{1}{k} \le t \le T$ then

$$\begin{aligned} \left|v^{(k)}(t+h) - v^{(k)}(t)\right| &= k\left| \int_{t-\frac{1}{k}+h}^{t+h} v(s)ds - \int_{t-\frac{1}{k}}^{t} v(s)ds \right| \\ &= k\left| \int_t^{t+h} v(s)ds - \int_{t-\frac{1}{k}}^{t-\frac{1}{k}+h} v(s)ds \right| \\ &\le k\left| \int_t^{t+h} v(s)ds \right| + k\left| \int_{t-\frac{1}{k}}^{t-\frac{1}{k}+h} v(s)ds \right| \\ &\le 2kv(T)h \end{aligned}$$

which implies the Lipschitz of $v^{(k)}(t)$.

Since $v(s)$ is a cad-lag process, $v^{(k)}(s)$ converges, for any fixed ω, to $v(s^-)$ for every s, as k approaches infinity. Moreover, except for a countable set in s, we have $v(s^-) = v(s)$. By integration by parts

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(using Lemma (3.5)), we have

$$\int_t^T \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)dv_i(s-t) = \int_0^{T-t} \exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)dv_i(s)$$

$$= v_i(T-t)\exp\Big(-\int_0^{T-t}\alpha(\lambda+t)d\lambda\Big) \tag{3.91}$$

$$+ \int_0^{T-t} v_i(s)d\Big(\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)\Big).$$

Since $v_i^{(k)}(s)$ converges pointwise to $v_i^-(s) := v_i(s^-)$, we imply $v_i^{(k)}(s)$ converges to $v_i(s)$ almost everywhere. Thus

$$\lim_{k\to\infty}\int_0^{T-t} v_i^{(k)}(s)d\Big(\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)\Big) = \int_0^{T-t} v_i(s)d\Big(\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)\Big)$$

and

$$\lim_{k\to\infty}\exp\Big(-\int_0^{T-t}\alpha(\lambda+t)d\lambda\Big)v_i^{(k)}(T-t) = \exp\Big(-\int_0^{T-t}\alpha(\lambda+t)d\lambda\Big)v_i^-(T-t)$$

$$\le \exp\Big(-\int_0^{s}\alpha(\lambda+t)d\lambda\Big)v_i(T-t) \tag{3.92}$$

where $v_i^{(k)}(0) = v_i(0) = 0$.

$$\lim_{k\to\infty}\int_0^{T-t}\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)dv_i^{(k)}(s) =$$

$$\lim_{k\to\infty}\Big[\exp\Big(-\int_0^{T-t}\alpha(\lambda+t)d\lambda\Big)v_i^{(k)}(T-t) - \int_0^{T-t} v_i^{(k)}(s)d\Big(\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)\Big)\Big]$$

$$\le \exp\Big(-\int_0^{T-t}\alpha(\lambda+t)d\lambda\Big)v_i((T-t)^-) - \int_0^{T-t} v_i(s)d\Big(\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)\Big) \tag{3.93}$$

$$\le \exp\Big(-\int_0^{T-t}\alpha(\lambda+t)d\lambda\Big)v_i(T-t) - \int_0^{T-t} v_i(s)d\Big(\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)\Big)$$

$$= \int_0^{T-t}\exp\Big(-\int_0^s \alpha(\lambda+t)d\lambda\Big)dv_i(s).$$

$$\lim_{k\to\infty}\mathbb{E}\Big\{\sum_{i=1}^n c_i\int_t^T \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)dv_i^{(k)}(s-t)\Big\} \le \mathbb{E}\Big\{\sum_{i=1}^n c_i\int_t^T \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)dv_i(s-t)\Big\}.$$

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Moreover,

$$\begin{aligned} &\mathbb{E}\Big\{\Big|\int_t^T f(y_{xt}^{(v)}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds - \int_t^T f(y_{xt}^{(v^{(k)})}(s),s)\exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)ds\Big|\Big\} \\ \leq\ & K\mathbb{E}\Big\{\int_t^T \Big|1+|y_{xt}^{(v)}(s)|^{m-1}+|y_{xt}^{(v^{(k)})}(s)|^{m-1}\Big|\cdot\Big|y_{xt}^{(v)}(s)-y_{xt}^{(v^{(k)})}(s)\Big|ds\Big\} \\ \to\ & 0 \text{ as } k\to\infty \text{ by using (3.89).} \end{aligned}$$

This fact implies

$$\lim_{k\to\infty} J_{xt}(v^{(k)}) \leq J_{xt}(v) \text{ as } n\to\infty. \tag{3.94}$$

Let us denote $V_0 = \cup\{V^\epsilon : \epsilon > 0\}$, then the above inequality implies that the optimal cost u can be represented by

$$u(x,t) = \inf\{J_{xt}(v) : v\in V_0\},\ x\in\mathbb{R}^n,\ t\in[0,T]. \tag{3.95}$$

For $\delta > 0$, there exists $\eta(t)$ with $0\leq\eta(t)\leq L,\ \forall t\in[0,T]$ such that for $v(t)=\int_0^t \eta(s)ds$,

$$u(x,t)\leq J_{xt}(v)\leq u(x,t)+\frac{\delta}{2}.$$

Let ϵ such that $\frac{2}{\epsilon} > \max\left\{L, \frac{\delta\alpha_0}{n}\right\}$, and let $\xi_i = \frac{\delta\alpha_0}{2n}$, then $(\eta,\xi)\ \in\ V_\epsilon$,

$$u(x,t)\leq J_{xt}(\eta,\xi)\leq u(x,t)+\delta.$$

Therefore $u^\epsilon(x,t)\longrightarrow u(x,t)$ as $\epsilon\longrightarrow 0_+$. □

In summary, we have

Theorem 3.2. *With the same assumptions as in Theorem 3.1, there exist positive constants* $0 < k \leq K,\ m\geq 1$, *the optimal cost* u^ϵ *given by* (3.78) *satisfies the following:*

$$\begin{cases} (i)\quad k|x^+|^m - K \leq u^\epsilon(x,t)\leq K(1+|x|^m), \\ (ii)\quad |u^\epsilon(x,t)-u^\epsilon(x',t)|\leq K(1+|x|^{m-1}+|x'|^{m-1})|x-x'|, \\ (iii)\quad |u^\epsilon(x,t)-u^\epsilon(x,t')|\leq K(1+|x|^m)|t-t'|, \\ \dfrac{\partial u^\epsilon}{\partial t}, \dfrac{\partial^2 u^\epsilon}{\partial x_i\partial x_j}\in L^\infty_{loc}(\mathbb{R}^n\times[0,T]),\ u^\epsilon \text{ is convex, and} \\ 0\leq \dfrac{\partial^2 u^\epsilon}{\partial x_i^2}(x,t)\leq K(1+|x|^q),\ \text{with } q=(m-2)^+,\ i=1,\cdots,n \\ \text{for every } (x,t),(x',t')\in\mathbb{R}^n\times[0,T]. \end{cases} \tag{3.96}$$

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Moreover, for each $x \in \mathbb{R}^n,\ t \in [0,T],\ u^\epsilon(x,t) \to u(x,t)$ *as* $\epsilon \to 0_+$. (3.97)

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CHAPTER 4 EXISTENCE AND UNIQUENESS OF THE OPTIMAL COST

4.1 Variational Formulation

Define

$$A_0 = -\sum_{i,j=1}^{n} a_{ij}(t)\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{n} g_i(t)\frac{\partial}{\partial x_i}. \quad (4.1)$$

Define $\pi(x,\lambda) = (\lambda + |x|^2)^{-p}$ for $x \in \mathbb{R}^n$, where $\lambda > 0$ and $p > 0$ are constants which can be chosen later. Define

$$H = \left\{\varphi : \varphi(1+|x|^2)^{-\frac{p}{2}} \in L^2(\mathbb{R}^n)\right\} \text{ with the norm } |\varphi|_H = |\varphi(1+|x|^2)^{-\frac{p}{2}}|_{L^2(\mathbb{R}^n)}. \quad (4.2)$$

$$V = \left\{\varphi \in H : \text{for } i = 1, \cdots, n, \ \frac{\partial \varphi}{\partial x_i} \text{ exists and } \frac{\partial \varphi}{\partial x_i}(1+|x|^2)^{-\frac{p}{2}} \in L^2(\mathbb{R}^n)\right\}, \quad (4.3)$$

where $\frac{\partial \varphi}{\partial x_i}$ denotes the generalized derivative. In V we use the norm

$$\|\varphi\|_V = \left[|\varphi|_H^2 + \sum_{i=1}^{n}\left|\frac{\partial \varphi}{\partial x_i}(1+|x|^2)^{-\frac{p}{2}}\right|_H^2\right]^{\frac{1}{2}}. \quad (4.4)$$

Note that the norms $|\cdot|_H$ and $\|\cdot\|_V$ can be defined, respectively, from the inner products

$$(w,z)_H = \int_{\mathbb{R}^n} w(x)z(x)(1+|x|^2)^{-p}dx, \quad \forall w, z \in H$$

and

$$(w,z)_V = \int_{\mathbb{R}^n} [w(x)z(x) + w'(x)z'(x)](1+|x|^2)^{-p}dx, \quad \forall w, z \in V.$$

It is then easy to prove that H and V are Hilbert spaces, with V continuously and densely embedded into H. Identifying H with its dual H' and the notation V' be the dual space of V we have $V \subset H = H' \subset V'$. For $v' \in V'$ and $v \in V$, denote the value of v' on v by $< v', v >$. The norm in V' is defined by

$$\|v'\|_{V'} = \sup_{v \in V, \|v\|_V \le 1} < v', v >, \ \forall v' \in V'.$$

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We denote by $\|\cdot\|_X$ the norm in the Hilbert space X. We call X' the dual space of X. We denote by $L^2(0,T;X)$ the Hilbert space of the real functions $f:(0,T)\to X$ measurable, such that

$$\|f\|_{L^2(0,T;X)} = \left(\int_0^T \|f(t)\|_X^2 dt\right)^{\frac{1}{2}} < \infty.$$

We denote by $\mathcal{D}'(]0,T[;V)$ the space of linear continuous mappings from $\mathcal{D}(]0,T[)\to V$, the space of distributions on $]0,T[$ with values in V. If $\varphi\in\mathcal{D}(]0,T[)$ and $f\in\mathcal{D}'(]0,T[;V)$, we have $f(\varphi)\in V$, and $\varphi\longrightarrow f(\varphi)$ is a continuous map of $\mathcal{D}(]0,T[)\longrightarrow V$. We define the derivative $\frac{df}{dt}\in\mathcal{D}'(]0,T[;V)$ by

$$\varphi\to\frac{df}{dt}(\varphi) = -f\left(\frac{d\varphi}{dt}\right).$$

This formula defines a continuous linear map from $\mathcal{D}(]0,T[)\longrightarrow V$. Hence

$$\frac{df}{dt}\in\mathcal{D}'(]0,T[;V).$$

$$f(\varphi) = \int_0^T f(t)\varphi(t)dt, \tag{4.5}$$

where the integral is the Lebesgue integral with values in V and $\varphi\longrightarrow f(\varphi)$ is a continuous map of $\mathcal{D}(]0,T[)\longrightarrow V$. In this manner, we define $\tilde{f}\in\mathcal{D}'(]0,T[;V)$ and a linear map $f\longrightarrow\tilde{f}$ of

$$L^2(0,T;V)\longrightarrow\mathcal{D}'(]0,T[;V)$$

which is a linear continuous injection. Hence, we identify $\tilde{f}$ with f and we have $\frac{df}{dt}\in D'(]0,T[;V)$.

$$\frac{df}{dt}\in\mathcal{D}'(]0,T[;V).$$

We then introduce the space

$$Z = \left\{f|f\in L^2(0,T;V);\frac{df}{dt}\in L^2(0,T;V')\right\}, \tag{4.6}$$

equipped with the norm

$$\|f\|_Z = \left(\int_0^T\left(\|f\|_V^2 + \left\|\frac{df}{dt}\right\|_{V'}^2\right)dt\right)^{\frac{1}{2}}. \tag{4.7}$$

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It is proved in [61] that all functions $f \in Z$ are, with eventual modification on a set of measure zero, continuous from $[0,T] \to H$. Abbreviating, we shall denote by $Z \subset C([0,T];H)$ the space of continuous functions from $[0,T] \to H$. For any $y \in \mathbb{R}^n$, define $B(y) = \sum_{i=1}^{n} \beta(y_i + c_i)$. Define

$$\begin{aligned} b(t;u,v) = \int\limits_{\mathbb{R}^n} \Big[\sum_{i,j=1}^{n} a_{ij}(t) \left(\frac{\partial u}{\partial x_i}(x)\right) \left(\frac{\partial v}{\partial x_j}(x) - 2px_j(1+|x|^2)^{-1}v(x)\right) \\ - \sum_{i=1}^{n} g_i(t) \left(\frac{\partial u}{\partial x_i}(x)\right) v(x) + \frac{1}{\epsilon} B(\nabla u)(x)v(x) + \alpha(t)u(x)v(x) \Big] (1+|x|^2)^{-p} dx. \end{aligned} \tag{4.8}$$

For any $\alpha = \alpha(t) > 0$ and $F \in L^2(0,T;V')$ then we say that $u \in Z$ is a weak solution of

$$\begin{cases} -\dfrac{du}{dt} + A_0 u + \dfrac{1}{\epsilon} B(\nabla u) + \alpha u = F, \\ u(\cdot,T) = 0, \end{cases}$$

if and only if for every $v \in V$ we have

$$\begin{cases} - < \dfrac{du}{dt}, v > \ + b(t;u(t),v) \ = < F, v >, \text{ a.e. } t \in]0,T[, \\ u(\cdot,T) = 0, \end{cases}$$

where $< \cdot, \cdot >$ denotes the duality bracket. One should remark that an element u on $L^2(0,T;V)$ such that u_t belongs to $L^2(0,T;V')$, then u can be regarded as a continuous function from $[0,T]$ into V. This makes clear the meaning of the boundary condition at $t = T$. To obtain the desired result, we need some auxiliary lemmas.

4.2 Auxiliary Lemmas

Lemma 4.1. *Suppose that* (3.13) *and* (3.12) *hold. Let* $\bar{\alpha}_0 = \max_{t \in [0,T]} \alpha(t)$, *there exists a large enough* $\bar{\alpha} > \bar{\alpha}_0$ *such that for every* $h \in L^2(0,T;V')$ *there exists a unique weak solution* $u \in Z$ *to the equation*

$$\begin{cases} -\dfrac{du}{dt} + A_0 u + \dfrac{1}{\epsilon} B(\nabla u) + \bar{\alpha} u = h, \\ u(\cdot,T) = 0. \end{cases} \tag{4.9}$$

Proof.

Proof of Uniqueness.

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We consider $A_1 u \equiv A_0 u + \frac{1}{\epsilon} B(\nabla u) + \bar{\alpha} u$. We have

$$\begin{aligned}
< A_1 u, v > = \int_{\mathbb{R}^n} \Big[\sum_{i,j=1}^{n} a_{ij}(t) \left(\frac{\partial u}{\partial x_i}(x) \right) \left(\frac{\partial v}{\partial x_j}(x) - 2p x_j (1+|x|^2)^{-1} v(x) \right) \\
- \sum_{i=1}^{n} g_i(t) \left(\frac{\partial u}{\partial x_i}(x) \right) v(x) + \frac{1}{\epsilon} B(\nabla u)(x) v(x) + \bar{\alpha} u(x) v(x) \Big] (1+|x|^2)^{-p} dx.
\end{aligned} \tag{4.10}$$

Since $a_{ij}(t)$, $g_i(t)$ are continuous and bounded functions, and a_{ij} satisfies the ellipticity, then we obtain from the Cauchy-Schwarz inequality that

$$< A_0 u - A_0 v, u - v > \geq k_1 \sum |u_{x_i} - v_{x_i}|_H^2 - k_2 |u - v|_H^2$$

for some positive k_1, k_2. Since B is Lipschitz, applying Cauchy-Schwarz inequality once again, we can obtain that

$$< \frac{1}{\epsilon} B(\nabla u) - \frac{1}{\epsilon} B(\nabla v), u - v > \geq -\frac{k_1}{2} \sum |u_{x_i} - v_{x_i}|_H^2 - k_3 |u - v|_H^2$$

for some $k_3 > 0$. Let $\bar{\alpha} > k_2 + k_3 + k_1/2$ we have

$$< A_1 u - A_1 v, u - v > \geq \frac{k_1}{2} \|u - v\|_V^2. \tag{4.11}$$

The uniqueness of the solution follows straightforward from the above coercive property.

Proof of Existence.

Since V is a separable Hilbert space, there exists a countable basic $w_1, w_2, \cdots, w_m, \cdots$ of V in the following sense:

$\forall m, w_1, \cdots, w_m$ are linearly independent and the linear combinations $\sum_{\text{finite}} \xi_j w_j,\ \xi_j \in \mathbb{R}$, are dense in V. We seek an approximate solution

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i,$$

where the $g_{im}(t)$ being solutions of the following system of differential equations:

$$\begin{cases} -\left\langle \dfrac{du_m}{dt}, w_i \right\rangle + b(t; u_m, w_i) = < h, w_i >, \ 1 \leq i \leq m, \\ u_m(T) = 0. \end{cases} \tag{4.12}$$

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Since $b(t; u, v)$ is linear with respect to v, we can multiply equations (4.12) by $g_{im}(t)$ and add up to have

$$-\left\langle \frac{d}{dt}u_m(t), u_m(t) \right\rangle + b(t; u_m(t), u_m(t)) =< h(t), u_m(t) >,$$

that is,

$$-\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_V^2 + b(t; u_m(t), u_m(t)) =< h(t), u_m(t) >,$$

so that by integrating between 0 and T and applying (4.11) with u replaced by u_m and v replaced by 0 we obtain

$$\begin{aligned}
\|u_m(0)\|_V^2 + k_1 \int_0^T \|u_m(t)\|_V^2 dt &\leq 2\int_0^T | < h(t), u_m(t) > | dt \\
&\leq 2\int_0^T \|h(t)\|_{V'} \|u_m(t)\|_V dt \\
&\leq \frac{k_1}{2}\int_0^T \|u_m(t)\|_V^2 dt + \frac{2}{k_1}\int_0^T \|h(t)\|_{V'}^2 dt.
\end{aligned}$$

From this we can deduce the estimate

$$\int_0^T \|u_m(t)\|_V^2 dt \leq C\left(\int_0^T \|h(t)\|_{V'}^2 dt\right) \tag{4.13}$$

for some positive constant C. Therefore u_m ranges in a bounded set in $L^2(0, T; V)$, by the Banach - Alaoglu theorem, we may extract a subsequence u_μ such that

$$u_\mu \rightarrow u \text{ weakly in } L^2(0, T; V). \tag{4.14}$$

Let j be fixed but arbitrary and let $\mu > j$. Then (4.12) is valid with $m = \mu$. Multiply both sides of (4.12) by $\varphi(t)$ where

$$\varphi(t) \in C^1[0, T], \quad \varphi(0) = 0, \tag{4.15}$$

and integrate over $(0, T)$. Setting $\varphi_j(t) = \varphi(t)w_j$, we have

$$\int_0^T [< u_\mu(t), \varphi_j'(t) > + b(t; u_\mu(t), \varphi_j(t))]dt = \int_0^T < h, \varphi_j(t) > dt + < u_{0\mu}, \varphi_j(0) > . \tag{4.16}$$

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We can then proceed to the limit as $\mu \to \infty$. This gives

$$\int_0^T [< u, \varphi_j' > + b(t; u, \varphi_j)] dt = \int_0^T < h, \varphi_j > dt + \ < u_0, \varphi_j(0) > . \tag{4.17}$$

But the above is true for any φ satisfying (4.15). Therefore, we may take $\varphi \in \mathcal{D}(]0, T[)$ and hence (4.17) gives

$$-\frac{d}{dt} < u(t), w_j > + b(t; u(t), w_j) =< h(t), w_j > \tag{4.18}$$

where the derivative is taken in $\mathcal{D}'(]0, T[)$. But in (4.18) j is arbitrary and since finite linear combinations of w_j are dense in V, we deduce

$$-\frac{du}{dt} + A_0 u + \frac{1}{\epsilon} B(\nabla u) + \bar{\alpha} u = h. \tag{4.19}$$

Therefore,

$$\frac{du}{dt} = A_0 u + \frac{1}{\epsilon} B(\nabla u) + \bar{\alpha} u - h \in L^2(0, T; V'),$$

and hence $u \in Z$. Moreover, it follows from (4.13) that

$$\left\| \frac{du}{dt} \right\|^2_{L^2(0,T;V')} \leq C \left(\int_0^T \|h(t)\|^2_{V'} \right) dt.$$

□

For any $\lambda > 0, \ q > 0, \ f : \mathbb{R}^n \times [0, T] \to \mathbb{R}$, define

$$\|f\|_{\lambda,q} = |f(x,t)(\lambda + |x|^2)^{-q}|_{L^\infty(\mathbb{R}^n \times [0,T])}. \tag{4.20}$$

For any $q > 0$, let Z_q be the set of all continuous functions $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ such that $f(x,t)(1 + |x|^2)^{-q} \to 0$ as $|x| \to \infty$ uniformly in t. Let φ be a mollification kernel, i.e., $\varphi \in C^\infty(\mathbb{R}^n)$, $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^n$, $\varphi \equiv 0$ for $|x| \geq 1$, and $\int_{\mathbb{R}^n} \varphi(x) dx = 1$. For each $k = 1, 2, \cdots$, and $x \in \mathbb{R}^n$, define

$$F_k(x,t) \ = \begin{cases} f(x,t) & \text{if } |x| \leq k, \\ 0 & \text{otherwise}, \end{cases} \tag{4.21}$$

and

$$f_k(x,t) = \int_{\mathbb{R}^n} k^n \varphi(k(x-y)) F_k(y,t) dy. \tag{4.22}$$

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Lemma 4.2. *Suppose $q > 0$ and $p > \frac{n}{2} + 2q$ and given any function $f \in Z_q$, then*

i) $f \in L^2(0, T; V')$ and

ii) if $f_k \in Z_q$, for $k = 1, 2, \cdots$, and for some $\lambda > 0$ we have $\|f_k - f\|_{\lambda,q} \to 0$ as $k \to \infty$, then $f_k \to f$ in $L^2(0, T; V')$ as $k \to \infty$ where f_k is defined by (4.22).

Proof. i) Let $f \in Z_q$ with $p > \dfrac{n}{2} + 2q$, there exists $C > 0$ such that $|f(x,t)| \leq C(1 + |x|^2)^q$. We will show that $\displaystyle\int\limits_{\mathbb{R}^n} \big(f(x,t)(1 + |x|^2)^{-\frac{p}{2}}\big)^2 dx$ is bounded uniformly in t. We have

$$\int\limits_{\mathbb{R}^n} \big(f(x,t)(1 + |x|^2)^{-\frac{p}{2}}\big)^2 dx \leq C \int\limits_{\mathbb{R}^n} (\lambda + |x|^2)^{-p+2q} dx.$$

Moreover, by calculation we obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} (\lambda + |x|^2)^{-p+2q} dx &= \int_0^\infty \left(\int_{\partial B(x_0,r)} (\lambda + |x|^2)^{-p+2q} dS \right) dr \\
&= n\alpha(n) \int_0^\infty (\lambda + |r|^2)^{-p+2q} \cdot r^{n-1} dr \\
&< n\alpha(n) \int_0^\infty r^{-2p+4q+n-1} dr
\end{aligned}$$

where $n\alpha(n)$ is surface area of unit sphere $\partial B(0, 1)$ in $\mathbb{R}^n$. Denote $a = p - 2q - \frac{n}{2}$ then $\displaystyle\int_0^\infty r^{-2p+4q+n-1} dr = \int_0^\infty r^{-2a-1}$ for $a > 0$. This integral converges. Therefore $f \in L^2(0, T; H) \subset L^2(0, T; V')$.

ii) Let $\varphi \in V$, $p > \dfrac{n}{2} + 2q$ then for each $t \in [0, T]$, by using Cauchy-Schwarz inequality and the

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above result, we have

$$\left|\int_{\mathbb{R}^n} \varphi\cdot(f_k-f)\cdot(1+|x|^2)^{-p}dx\right| \leq \int_{\mathbb{R}^n} |\varphi|\cdot|f_k-f|\cdot(1+|x|^2)^{-p}dx$$
$$= \int_{\mathbb{R}^n} |\varphi|\cdot(1+|x|^2)^{-\frac{p}{2}}\cdot|f_k-f|\cdot(1+|x|^2)^{-\frac{p}{2}}dx$$
$$\leq \Big(\int_{\mathbb{R}^n} \varphi^2\cdot(1+|x|^2)^{-p}dx\Big)^{\frac{1}{2}}\cdot\Big(\int_{\mathbb{R}^n} |f_k-f|^2\cdot(1+|x|^2)^{-p}dx\Big)^{\frac{1}{2}} \tag{4.23}$$
$$< \|\varphi\|_V\cdot\Big(\int_{\mathbb{R}^n} |f_k-f|^2\cdot(1+|x|^2)^{-2q}\cdot(1+|x|^2)^{-p+2q}dx\Big)^{\frac{1}{2}}$$
$$= \|f_k-f\|_{\lambda,q}\cdot\|\varphi\|_V\cdot\int_{\mathbb{R}^n}(1+|x|^2)^{-p+2q}dx$$
$$= K\|f_k-f\|_{\lambda,q}\cdot\|\varphi\|_V.$$

From inequality (4.23) we have $|f_k - f|_{V'} \leq K\|f_k - f\|_{\mu,q}$ uniformly in t for some constants $K > 0$.

□

Lemma 4.3. *Let $q > 0$, $f \in Z_q$, $f_k, k = 1, 2, \cdots$, be defined by (4.22), then we have*

i) $f_k \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ for $k = 1, 2, \cdots$,

ii) For every $x \in \mathbb{R}^n, t \in [0, T]$, $\lim_{k\to\infty} f_k(x, t) = f(x, t)$, the convergence being uniform on any compact set,

iii) For every constant $\lambda > 0$, $\|f_k - f\|_{\lambda,q} \to 0$ as $k \to \infty$,

iv) For every constant $\lambda > 0$, $\lim_{k\to\infty} \|f_k\|_{\lambda,q} = \|f\|_{\lambda,q}$.

Proof. i) Continuity of f_k follows from the continuity of φ. Differentiation can be carried under the integral sign, so that the differentiability can be carried under the integral sign, so that the differentiability properties of f_k follow from those of φ. Since the support of F_k is contained in a compact subset of $\mathbb{R}^n \times \mathbb{R}$, then we have $f_k \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$. ii)

$$|f_k(x,t) - f(x,t)| \leq \int_{\mathbb{R}^n} k^n\varphi(k(x-y))|F_k(y,t) - F_k(x,t)|dy$$
$$\leq \sup_{\|y-x\|\leq\varepsilon} |F_k(y,t) - F_k(x,t)|. \tag{4.24}$$

Since the last term tends to zero with ε at each continuous point (x, t), the convergence to zero

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being uniform for any compact set of continuity points, then ii) follows. iii) The proof of iii) is similar to ii). iv) Once iii) has been proved then iv) follows immediately. □

Lemma 4.4. *For $i = 1, 2$, let $F_i \in Z_q$, $q > 0$, and $u_i \in C^{2,1}(\mathbb{R}^n \times [0, T]) \cap Z_q$. Let ϵ and α^* be positive constants. For $i = 1, 2$, then let u_i be the weak solutions of*

$$\begin{cases} -\dfrac{\partial u}{\partial t} + A_0 u_i + \dfrac{1}{\epsilon} B(\nabla u_i) + \alpha^* u_i = F_i, \\ u_i(\cdot, T) = 0. \end{cases} \tag{4.25}$$

Then for every η with $0 < \eta < \alpha^*$ there is a $\lambda_0 > 0$ such that for $\lambda > \lambda_0$,

$$(\alpha^* - \eta)\|u_1 - u_2\|_{\lambda,q} \le \|F_1 - F_2\|_{\lambda,q},$$

where λ_0 depends on n, the coefficients of A_0, q, ϵ, the Lipschitz constant of B, and η.

Proof. From (4.25) we have

$$A_0(u_1 - u_2) + \frac{1}{\epsilon}(B(\nabla u_1) - B(\nabla u_2)) + \alpha^*(u_1 - u_2) = F_1 - F_2 + \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} \tag{4.26}$$

implies

$$\begin{aligned} A_0(u_1 - u_2) = & \quad -\eta(u_1 - u_2) + \epsilon^{-1}(B(\nabla u_2) - B(\nabla u_1)) \\ + & \quad F_1 - F_2 - (\alpha^* - \eta)(u_1 - u_2) + \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}. \end{aligned} \tag{4.27}$$

Set $W(x, t) = u_1 - u_2$, $w(x, t) = W(x, t)\pi(x, \lambda)$ then $w(x, t) \to 0$ as $|x| \to +\infty$ uniformly in t. Suppose $w(x, t) \neq 0$ and $w(x, t) > 0$ for some $x \in \mathbb{R}^n$, $t \in [0, T]$. If $w(x, t) < 0$ then consider $w(x, t) = u_2 - u_1$. Then there exists (x_0, t_0) such that

$$w(x_0, t_0) = \max_{(x_0,t_0)\in\mathbb{R}^n\times[0,T]} w(x, t) > 0. \tag{4.28}$$

Note that $u_1(x, T) = u_2(x, T) = 0$ then $W(x, T) = 0$.

By calculation we obtain

$$A_0\pi(x, \lambda) + \frac{[\mathrm{tr}(\sigma\sigma^*\nabla\pi(x, \lambda)\nabla\pi(x, \lambda)^*)]}{\pi(x, \lambda)} = \delta(x, \lambda)\pi(x, \lambda) \tag{4.29}$$

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with $\sup_x|\delta(x,\lambda)| \to 0$ as $\lambda \to \infty$. Since β is a non increasing, Lipschitz function, then we have

$$\begin{aligned} B(\nabla u_2) - B(\nabla u_1) &= \sum_{i=1}^{n} [\beta(u_{2x_i} + c_i) - \beta(u_{1x_i} + c_i)] \\ &= \sum_{i=1}^{n} \gamma_i \cdot (u_{2x_i} - u_{1x_i}) \text{ (where } |\gamma_i| \le 1) \\ &= \gamma \cdot (\nabla u_2 - \nabla u_1) \text{ for some } \gamma \text{ depending on } x, t. \end{aligned} \tag{4.30}$$

By (4.28), we have $\nabla w(x_0, t_0) = 0$. Therefore

$$\nabla W(x_0, t_0)\pi(x_0, \lambda) = -W(x_0, t_0)\nabla\pi(x_0, \lambda). \tag{4.31}$$

Hence

$$\nabla W(x_0, t_0) = -W(x_0, t_0)\nabla\pi(x_0, \lambda)/\pi(x_0, \lambda). \tag{4.32}$$

Since $\nabla\pi(x_0, \lambda)/\pi(x_0, \lambda) \to 0$ as $\lambda \to \infty$, we obtain

$$B(\nabla u_2(x_0, t_0)) - B(\nabla u_1(x_0, t_0)) = W(x_0, t_0)\hat{\delta}(x, t, \lambda) \tag{4.33}$$

where $\sup_{(x,t)\in R^n\times[0,T]} |\hat{\delta}(x, t, \lambda)| \to 0$ as $\lambda \to \infty$. Also,

$$\begin{aligned} A_0 w(x_0, t_0) = A_0 W(x_0, t_0)\pi(x_0, \lambda) + W(x_0, t_0) A_0\pi(x_0, \lambda) \\ - \operatorname{tr}(\sigma\sigma^* \nabla W(x_0, t_0)\nabla\pi(x, \lambda)). \end{aligned} \tag{4.34}$$

Applying to the first term in the right hand side of (4.34) equality (4.27), and then applying (4.33), (4.29), (4.32) we have

$$\begin{aligned} A_0 w(x_0, t_0) = (&-\eta W(x_0, t_0) + \epsilon^{-1} W(x_0, t_0)\hat{\delta}(x_0, t_0, \lambda))\pi(x_0, \lambda) \\ &+ W(x_0, t_0) A_0\pi(x_0, \lambda) - (\alpha^* - \eta)(u_1 - u_2) + (F_1 - F_2) + \frac{\partial w}{\partial t}(x_0, t_0) \\ &- \operatorname{tr}[\sigma\sigma^*(-\nabla\pi(x_0, \lambda))W(x_0, t_0)\nabla\pi(x_0, \lambda)]/\pi(x_0, \lambda) \\ &= W(x_0, t_0)\pi(x_0, \lambda)(-\eta + \epsilon^{-1}\hat{\delta}(x_0, t_0, \lambda) + \delta(x_0, \lambda)) \\ &+ F_1 - F_2 - (\alpha^* - \eta)(u_1 - u_2) + \frac{\partial w}{\partial t}(x_0, t_0). \end{aligned} \tag{4.35}$$

Choose λ such that $-\eta + \epsilon^{-1}\hat{\delta}(x_0, t_0, \lambda) + \delta(x_0, \lambda) < 0$. Since $A_0 w(x_0, t_0) \ge 0$ and $\frac{\partial w}{\partial t}(x_0, t_0) \le 0$ then we have

$$\begin{aligned} (F_1 - F_2)(x_0, t_0) \ge \quad & (\alpha^* - \eta)(u_1(x_0, t_0) - u_2(x_0, t_0)) \\ = \quad & \sup_{(x,t)\in\mathbb{R}^n\times[0,T]} (u_1(x, t) - u_2(x, t)). \end{aligned} \tag{4.36}$$

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Since u_1, u_2 have the same role, then we have

$$(F_2 - F_1)(x_0', t_0') \geq (\alpha^* - \eta) \sup_{(x,t)\in\mathbb{R}^n\times[0,T]} (u_2(x,t) - u_1(x,t)) \tag{4.37}$$

if $F_1 - F_2$ attains its maximum at x_0', t_0'. From (4.36), (4.37), the desired result follows. □

Denote

- $C^{2,1}(Q)$ is the set of bounded continuous functions $u(x,t), (x,t) \in Q$, such that their derivatives u_x, u_{xx}, u_t are bounded and continuous in Q.

- $C^{\mu,\mu/2}(Q)$ is the Banach space of function $u(x,t)$ bounded Holder continuous with exponent μ in x and $\mu/2$ in t, where $\mu \in (0,1)$. In other words,

$$|u|_Q^{(\mu)} := \max_{(x,t)\in Q} |u(x,t)| + \sup_{(x,t),(x',t)\in Q} \frac{|u(x,t) - u(x',t)|}{|x - x'|^\mu} + \sup_{(x,t),(x,t')\in Q} \frac{|u(x,t') - u(x,t')|}{|t - t'|^{\mu/2}} < \infty.$$

- $C^{2+\mu,1+\mu/2}(Q)$ is the space of $u(x,t)$ bounded continuous together with u_t, u_x, u_{xx} and having the following finite norm:

$$|u|_Q^{(2+\mu)} := \max_{(x,t)\in Q} |u(x,t)| + \sum_{i=1}^n \max_{(x,t)\in Q} |u_{x_i}(x,t)| + |u_t|_Q^{(\mu)} + \sum_{i,j=1}^n |u_{x_i x_j}|_Q^{(\mu)}.$$

If $u(x,t)$ belongs to $C^{\mu,\mu/2}(Q)$ (reps. $C^{2+\mu,1+\mu/2}(Q)$) for any bounded cylinder, we say that $u(x,t)$ belongs to $C_{loc}^{\mu,\mu/2}(\mathbb{R}^n \times [0,T])$ (reps. $C_{loc}^{2+\mu,1+\mu/2}(\mathbb{R}^n \times [0,T])$).

Lemma 4.5. *Let $\epsilon > 0$, $p > 0$ with $p > \frac{n}{2} + 2q$ then there exists $\lambda_1 > 0$ such that for $\lambda > \lambda_1$, if $f \in Z_q$ and if u is the unique weak solution in $L^2(0,T;V)$ guaranteed by Lemma 4.1 and Lemma 4.2 to*

$$\begin{cases} -\dfrac{\partial u}{\partial t} + A_0 u + \dfrac{1}{\epsilon} B(\nabla u) + \bar{\alpha} u = f, \\ u(\cdot, T) = 0 \end{cases}$$

then $\frac{\bar{\alpha}}{2}\|u\|_{\lambda,q} \leq \|f\|_{\lambda,q}$. Moreover, if $f \in C_{loc}^{\mu,\mu/2}(\mathbb{R}^n \times [0,T])$ then $u \in C_{loc}^{2+\mu,1+\mu/2}(\mathbb{R}^n \times [0,T])$.

Proof. Let $f_k, k = 1, 2, \cdots$ be defined as in Lemma 4.2. Since f_k is smooth, by Ladyzhenskaya *et al.* [58, Theorem 8.1, Chapter V] , there is a solution u_k to $-\frac{\partial u}{\partial t} + A_0 u_k + \frac{1}{\epsilon} B(\nabla u_k) + \bar{\alpha} u_k = f_k$

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belonging to $C^{2+\mu,1+\mu/2}(\mathbb{R}^n \times [0,T])$ for any $\mu \in (0,1)$. Hence $u_k \in L^2(0,T;V)$. By Lemma 4.1, u_k is unique in $L^2(0,T;V)$. Moreover, $u_k \in Z_q$. Apply Lemma 4.4 with $\alpha^* = \bar{\alpha}, F_1 = f_k, F_2 = 0$. We have

$$\frac{\bar{\alpha}}{2}\|u_k\|_{\lambda,q} \leq \|f_k\|_{\lambda,q}. \tag{4.38}$$

By Lemma 4.3(iii), we have $\|f_k - f\|_{\lambda,q} \to 0$ as $k \to \infty$. Then by Lemma 4.2(ii), $f_k \to f$ in $L^2(0,T;V')$. By the continuity part of the statement of Lemma 4.1, $u_k \to u$ in $L^2(0,T;V)$ which implies that there is a subsequence of $\{u_k\}$ converges almost every where to u in $L^2(0,T;V)$. Using Lemma 4.3(iv), taking $k \to \infty$ in (4.38) for this subsequence , we have $\frac{\bar{\alpha}}{2}\|u\|_{\lambda,q} \leq \|f\|_{\lambda,q}$. Recall that $\{f_k\}$ are uniformly bounded on each compact set by Lemma 4.2. Moreover, if on a bounded cylinder Q, f is Hölder continuous with exponent μ, we can also show that $|f_k|^{(\mu)}(Q)$ are uniformly bounded. This fact can be proved similarly to Lemma 4.2 (see Agmon [1, Theorems 1.5, 1.7 and 1.8] for the idea of the proof). By arguments in the proof of Ladyzhenskaya *et al.* [58, Theorem 8.1, Chapter V] and the results in [58, Chapter V] (mostly in the formulation of Theorems 5.4 and 6.1), we claim that $|u_k|_Q^{(2+\mu)} < c(Q)$ which does not depend on k. Employing a usual diagonal process, we can extract from u_k a subsequence that converges together with the derivatives $u_{k_x}, u_{k_{xx}}$ and u_{k_t} at each point of $\mathbb{R}^n \times [0,T]$ to some function $\hat{u}$ and its corresponding derivatives. Since u_k converges to u in $L^2(0,T,V)$, we must have $\hat{u} \equiv u$. Clearly we have $|u|_Q^{(2+\mu)} < c(Q)$. As a result, $u \in C_{loc}^{2+\mu,1+\mu/2}(\mathbb{R}^n \times [0,T])$. □

4.3 The Regularity Of The Solution

Theorem 4.1. *Let $\epsilon > 0$ and let $\alpha(t) > 0$ be the discount factor. Let f satisfies* (3.13)*. Then the penalty equation* (3.73) *has a weak solution $u^\epsilon \in Z_q$. This weak solution is unique among all continuous functions of at most polynomial growth (i.e., function in $Z_{q'}$ for some $q' > 0$). Moreover, $u^\epsilon \in C_{loc}^{2+\mu,1+\mu/2}(\mathbb{R}^n \times [0,T])$ for every $\mu \in (0,1)$.*

Proof. Suppose that $p > \frac{n}{2} + 2q$, and q' with $\frac{m}{2} < q' < q$. Note that $f \in Z_{q'}$. For any $u \in Z_{q'}$,

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define $U = T_f u$ be the weak solution of

$$\begin{cases} -\dfrac{\partial U}{\partial t} + A_0 U + \dfrac{1}{\epsilon} B(\nabla U) + \bar{\alpha} U = (\bar{\alpha} - \alpha(t))u + f, \\ U(\cdot, T) = 0 \end{cases}$$

guaranteed by Lemma 4.1 and Lemma 4.2. For u_1 and u_2 in $Z_{q'}$, let $U_1 = T_f u_1$ and $U_2 = T_f u_2$. Let $F_1 = (\bar{\alpha} - \alpha(t))u_1 + f$ and $F_2 = (\bar{\alpha} - \alpha(t))u_2 + f$, where $\bar{\alpha} - \alpha(t) > 0$, $\forall t \in [0, T]$. By Lemma 4.5, there is $\lambda > 0$ such that $\|U_i\|_{\lambda, q'}, i = 1, 2$ are finite. As a direct consequence, $U_i(x, t)(1 + x^2)^{-q} \longrightarrow 0$ as $x \longrightarrow \infty$ uniformly in t. Moreover, the continuity of U_i are guaranteed by [58, Theorem 1.1, Chapter V]. Hence, $U_i \in Z_q$ for $i = 1, 2$. Applying Lemma 4.4, for $0 < \eta < \alpha_0$, where $\alpha_0 = \min_{t \in [0,T]} \alpha(t)$, there is λ_0 sufficiently large such that

$$\begin{aligned} (\bar{\alpha} - \eta)\|U_1 - U_2\|_{\lambda_0, q} &\leq \|F_1 - F_2\|_{\lambda_0, q} \\ &= \|(\bar{\alpha} - \alpha(t))(u_1 - u_2)\|_{\lambda_0, q} \\ &< (\bar{\alpha} - \alpha_0)\|(u_1 - u_2)\|_{\lambda_0, q}. \end{aligned} \tag{4.39}$$

Note that (4.39) shows that T_f is a contraction map in $\| \cdot \|_{\lambda_0, q}$ norm with contraction constant $(\bar{\alpha} - \alpha_0)(\bar{\alpha}_0 - \eta)^{-1} < 1$. By the last statement of Lemma 4.5, λ_0 can be chosen to be the same for all $m/2 < q' < q$. It is also noted that although in Lemma 4.4 we require solutions U_1 and U_2 to belongs to $C^{2,1}(\mathbb{R}^n \times [0, T])$, we claim that (4.39) holds for any $U_1, U_2 \in Z_q$ by limitation arguments as in Lemma 4.5. These arguments will also be used again in the following part of this proof. Since any weak solution of (3.73) in some $Z_{q'}$ space is a fixed point of T_f, this proves the uniqueness part of the theorem. Suppose that $p = n + m$, $q < \dfrac{m}{2} + \dfrac{n}{4}$ and we have $Z_r \subset Z_s$ for $0 < r < s$. Now we will prove that T_f is a contraction map from Z_q into Z_q. Assume $u_1, u_2 \in Z_q$, we will prove that (4.39) still hold. By using Lemma 4.3, there exist sequences $\{u_{1,k}\}_{k=1}^{\infty}$, $\{u_{2,k}\}_{k=1}^{\infty}$ in $C_0^{\infty}(\mathbb{R}^n \times \mathbb{R})$ which converge in $\| \cdot \|_{\lambda_0, q}$ norm to u_1 and u_2 respectively. Since $(\bar{\alpha} - \alpha(t))u_{i,k} + f \to (\bar{\alpha} - \alpha(t))u_i + f$ for $i = 1, 2$ in $\| \cdot \|_{\lambda_0, q}$ norm as $k \to \infty$, by Lemma 4.2 this convergence also in $L^2(0, T; V')$. Therefore by Lemma 4.1, $T_f u_{i,k} \to T_f u_i$ in $L^2(0, T; V)$, so there exist a subsequence of $\{T_f u_{i,k}\}_{k=1}^{\infty}$ converges almost every where to $T_f u_i$, then

$$\|T_f u_1 - T_f u_2\|_{\lambda_0, q} \leq \frac{(\bar{\alpha} - \alpha_0)}{(\bar{\alpha} - \eta)} \|(u_1 - u_2)\|_{\lambda_0, q}, \quad \forall u_1, u_2 \in Z_q. \tag{4.40}$$

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Note that $u \in Z_q$ if and only if $\|u\|_{\lambda_0,q} \longrightarrow 0$ as $\lambda_0 \longrightarrow \infty$. Hence, the above estimate shows that $T_f u_1 - T_f u_2 \in Z_q$. Similarly, we can show $T_f u_i \in Z_q$ for $i = 1, 2$. Let u be the unique fixed point of T_f in Z_q, then u is a weak solution of (3.73). Since $u \in Z_q$ is the solution to

$$\begin{cases} -\dfrac{\partial U}{\partial t} + A_0 U + \dfrac{1}{\epsilon} B(\nabla U) + \bar{\alpha}_0 U = (\bar{\alpha}_0 - \alpha(t))u + f, \\ U(\cdot, T) = 0. \end{cases}$$

By [58, Theorems 1.1, Chapter 5] in Ladyzhenskaya *et al.*, we have $u \in C^{\mu,\mu/2}_{loc}(\mathbb{R}^n \times [0,T])$ for some $\mu \in (0,1)$. Applying the last statement of Lemma 4.5, we claim that $u \in C^{2+\mu,1+\mu/2}_{loc}(\mathbb{R}^n \times [0,T])$. Since $f(x,t)$ is differentiable, we can apply Lemma 4.5 again to show that $u \in C^{2+\mu,1+\mu/2}_{loc}(\mathbb{R}^n \times [0,T])$ for any $\mu \in (0,1)$. □

Theorem 4.2. *Suppose that the condition* (3.13) *holds. Then the optimal cost* u^ϵ *given by* (3.78) *is the unique solution among continuous functions with at most polynomial growth to the Hamilton Jacobi Bellman equation* (3.73). *Moreover, for every* $\mu \in (0,1)$, $u^\epsilon \in C^{2+\mu,1+\mu/2}_{loc}(\mathbb{R}^n \times [0,T])$.

Proof. Suppose u^ϵ is the solution of (3.73). Let $h(s) = \exp\Big(-\int_t^s \alpha(\lambda)d\lambda\Big)$. Applying Ito's formula to $u^\epsilon(y_{xt}(s), s)h(s)$ we have

$$\begin{aligned} \mathbb{E}u^\epsilon(y_{xt}(T),T)h(T) &= \mathbb{E}\Big\{u^\epsilon(y_{xt}(t),t) + \int_t^T (-Au^\epsilon)(y_{xt}(s),s)h(s)ds \\ &+ \int_t^T \eta(t)\nabla u^\epsilon(y_{xt}(s),s)h(s)ds\Big\}. \end{aligned}$$

Since $u^\epsilon(y_{xt}(T),T) = 0$ and $y_{xt}(t) = x$ then

$$\begin{aligned} u^\epsilon(x,t) &= \mathbb{E}\Big\{\int_t^T Au^\epsilon(y_{xt}(s),s)h(s)ds\Big\} - \sum_{i=1}^n \mathbb{E}\Big\{\int_t^T \eta_i(t)\frac{\partial u^\epsilon}{\partial x_i}(y_{xt}(s),s)h(s)ds\Big\} \\ &= \mathbb{E}\Big\{\int_t^T f(y_{xt}(s),s)h(s)ds\Big\} - \sum_{i=1}^n \mathbb{E}\Big\{\int_t^T \frac{1}{\epsilon}\beta(\frac{\partial u^\epsilon}{\partial x_i}(y_{xt}(s),s) + c_i)h(s)ds\Big\} \\ &-\mathbb{E}\Big\{\int_t^T \eta_i(s)\frac{\partial u^\epsilon}{\partial x_i}(y_{xt}(s),s)h(s)ds\Big\}. \end{aligned}$$

Note that

$$-s\eta_i(t) - \frac{1}{\epsilon}\beta(s) \le \xi_i(t),$$

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then

$$-\frac{1}{\epsilon}\beta(\frac{\partial u^{\epsilon}}{\partial x_i}(y_{xt}(s),s)+c_i)-\eta_i(s)\frac{\partial u^{\epsilon}}{\partial x_i}(y_{xt}(s),s)\leq \xi_i(s)+c_i\eta_i(s).$$

Thus

$$\begin{aligned} u^{\epsilon}(x,t) &\leq \mathbb{E}\Big\{\int_t^T f(y_{xt}(s),s)h(s)ds+\sum_{i=1}^n\int_t^T c_ih(s)ds+\int_t^T \xi_i(s)h(s)ds\Big\} \\ &= J_{xt}(\eta,\xi). \end{aligned} \tag{4.41}$$

Now define feedback control $\hat{\eta}(y)=(\hat{\eta}_1(y),\cdots,\hat{\eta}_n(y))$ and $\hat{\xi}(y)=(\hat{\xi}_1(y),\cdots,\hat{\xi}_n(y))$ by

$$\begin{cases} \hat{\eta}_i(y)=-\frac{1}{\epsilon}\beta'\left(\frac{\partial u^{\epsilon}}{\partial x_i}(y_{xt}(s),s)+c_i\right), \\ \hat{\xi}_i(y)=-\frac{1}{\epsilon}\beta'\left(\frac{\partial u^{\epsilon}}{\partial x_i}(y_{xt}(s),s)+c_i\right)\left(\frac{\partial u^{\epsilon}}{\partial x_i}(y_{xt}(s),s)+c_i\right)-\frac{1}{\epsilon}\beta\left(\frac{\partial u^{\epsilon}}{\partial x_i}(y_{xt}(s),s)+c_i\right), \end{cases} \tag{4.42}$$

where $y_{xt}(s)$ is the solution to

$$\begin{cases} d(\hat{y}_{xt}(s))=(g+\hat{\eta}(y_{xt}(s)))ds+\sigma d\omega(s-t), \quad s>t, \\ \hat{y}_{xt}(0)=x. \end{cases}$$

Define $\hat{\eta}_i(s)=\hat{\eta}_i(\hat{y}_{xt}(s))$ and $\hat{\xi}_i(s)=\hat{\xi}_i(\hat{y}_{xt}(s))$. It is easy to check that $(\hat{\eta}(t),\hat{\xi}(t))\in V_{\epsilon}$. Moreover, we have

$$-\hat{\eta}_i(s)\left(\frac{\partial u^{\epsilon}}{\partial x_i}(\hat{y}_{xt}(s),s)\right)-\frac{1}{\epsilon}\beta\left(\frac{\partial u^{\epsilon}}{\partial x_i}(\hat{y}_{xt}(s),s)+c_i\right)=\hat{\xi}_i(s)+c_i\hat{\eta}_i(s)$$

which implies that the equality in (4.41) holds for this control. As a result,

$$u^{\epsilon}(x,t)=J_{xt}(\hat{\eta},\hat{\xi})$$

for this control. It completes the proof. □

Theorem 4.3. *Let the assumptions hold as in Theorem 3.1. Fix p with $n<p<\infty$. Let $\Omega\subset\mathbb{R}^n$ be an open ball. Denote $Q:=\Omega\times[0,T]$, there exists a sequence $\{\epsilon_k\}_{k=1}^{\infty}$ with $\epsilon_k\to 0_+$ as $k\to\infty$ such that for $1\leq i,j\leq n$*

$$u_{\epsilon_k}\to u \quad \textit{and} \quad \frac{\partial u_{\epsilon_k}}{\partial x_i}\to\frac{\partial u}{\partial x_i} \ \textit{uniformly on } \bar{Q},$$

$$\frac{\partial^2 u_{\epsilon_k}}{\partial x_i\partial x_j}\to\frac{\partial^2 u}{\partial x_i\partial x_j} \ \textit{weakly in } L^p(Q) \textit{ as } k\to\infty$$

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and

$$\frac{\partial u_{\epsilon_k}}{\partial t} \to \frac{\partial u}{\partial t} \text{ weakly in } L^p(Q) \text{ as } k \to \infty.$$

Proof. By the proof of Theorem 3.2, there exists a $K_1 > 0$ such that $|u_{\epsilon_k}| \le K_1, \left|\frac{\partial u_{\epsilon_k}}{\partial x_i}\right| \le K_1, \left|\frac{\partial^2 u_{\epsilon_k}}{\partial x_i \partial x_j}\right| \le K_1, \left|\frac{\partial u_{\epsilon_k}}{\partial t}\right| \le K_1$, on Q for $1 \le i, j \le n$. Since $W^{2,1,p}(Q)$ is reflexive, there is a sequence $\{\epsilon_k\}_{k=1}^{\infty}$ with $\epsilon_k \to 0_+$ as $k \to \infty$ such that u_{ϵ_k} converges weakly in $W^{2,1,p}(Q)$. By Theorem 3.2, we have $u_{\epsilon_k} \to u$ pointwise and the weak limits are unique, $u_{\epsilon_k} \to u$ weakly in $W^{2,1,p}(Q)$. Since $p > n$, then using Rellich-Kondrachov Theorem the embedding map $W^{2,1,p}(Q) \to C^{1,0}(\bar{Q})$ is compact. Therefore $u_{\epsilon_k} \to u$ and $\frac{\partial u_{\epsilon_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ uniformly on $\bar{Q}$ as $k \to \infty$. □

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CHAPTER 5 REGULARITY OF THE FREE BOUNDARY

Theorem 5.1. *Let the assumptions of Theorem 3.1 be satisfied. Then for* $i = 1, \cdots, n$ *there exists a real valued function* $\psi_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t) = \inf\Big\{x_i : \dfrac{\partial u}{\partial x_i}(x,t) + c_i > 0\Big\}$ *such that*

$$\frac{\partial u}{\partial x_i}(x,t) + c_i = 0 \quad \textit{if} \quad x_i \le \psi_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t) \tag{5.1}$$

and

$$\frac{\partial u}{\partial x_i}(x,t) + c_i > 0 \quad \textit{if} \quad x_i > \psi_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t) \tag{5.2}$$

for each $(x,t) = (x_1, \cdots, x_n, t) \in \mathbb{R}^n \times [0,T]$.

Proof. From (3.73) we have

$$Au^\epsilon - f = -\frac{1}{\epsilon}\sum_{i=1}^n \beta\Big(\frac{\partial u^\epsilon}{\partial x_i} + c_i\Big) \le 0.$$

Letting $\epsilon \to 0$, we get $Au \le f$. We also have

$$\sum_{i=1}^n \beta\Big(\frac{\partial u^\epsilon}{\partial x_i} + c_i\Big) = -\epsilon(Au^\epsilon - f).$$

Letting $\epsilon \to 0$, we get

$$\sum_{i=1}^n \beta\Big(\frac{\partial u}{\partial x_i} + c_i\Big) = \lim_{\epsilon\to 0}\sum_{i=1}^n \beta\Big(\frac{\partial u^\epsilon}{\partial x_i} + c_i\Big) = \lim_{\epsilon\to 0}(-\epsilon(Au - f)) = 0.$$

From the definition of β, we have $\dfrac{\partial u}{\partial x_i} + c_i \ge 0$. Since u is convex , $\dfrac{\partial u}{\partial x_i}$ is nondecreasing in x_i, hence if $x_i \le \psi_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t)$ then $\dfrac{\partial u}{\partial x_i} + c_i = 0$ and if $x_i > \psi_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t)$ then $\dfrac{\partial u}{\partial x_i} + c_i > 0$. □

Definition 5.2. *For any* i *with* $i = 1, \cdots, n$*, define*

$$\varphi_i = \{(x,t) \in \mathbb{R}^n \times [0,T] : \frac{\partial u}{\partial x_j}(x,t) + c_j > 0 \textit{ for all } j \ne i\} \tag{5.3}$$

and

$$\mathcal{F}_i = \varphi_i \cap \{(x,t) \in \mathbb{R}^n \times [0,T] : x_i = \psi_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, t)\}. \tag{5.4}$$

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The free boundary is $\overline{\varphi_1 \cap \cdots \cap \varphi_n} \setminus (\varphi_1 \cap \cdots \cap \varphi_n)$. Then we will show $\mathcal{F}_i$ is regular for each $i = 1, \cdots, n$. The others free boundary points are corner points. By symmetry, it clearly suffices to study the regularity of $\mathcal{F}_n$. The results will be done in the following. First, we will consider the bilinear form $a(t; u, v)$ associate with operator A define by (1.8)

$$a(t; u, v) = \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij}(t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} - \sum_{i=1}^n g_i(t) \frac{\partial u}{\partial x_i} u + \alpha(t) uv \Big\} dx, \tag{5.5}$$

Lemma 5.1. *Let $a(t; u, v)$ be defined by (5.5). Then $a(t; u, v)$ is coercive on $H_0^1(Q)$.*

Proof. For $u \in H_0^1(Q)$, we have

$$\int_\Omega u \frac{\partial u}{\partial x_i} dx = 0 \quad \text{for} \quad i = 1, \cdots, n,$$

and

$$a(t; u, u) = \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij}(t) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \sum_{i=1}^n g_i(t) \frac{\partial u}{\partial x_i} u + \alpha(t) u^2 \Big\} dx. \tag{5.6}$$

Since $\alpha(t) > 0$ and the ellipticity $\sum_{i,j=1}^n a_{ij}(t) \xi_i \xi_j \geq \lambda |\xi|^2$, we have $a(t; u, u) \geq \lambda_0 \int_\Omega (|\nabla u|^2 + u^2) dx$ $(\lambda_0 > 0)$ which implies $a(t; u, v)$ is coercive. □

Recall the problem in which we are interested in as follow: we seek a function u, in a suitable space, such that

$$\begin{cases} Au \ \leq \ f, \quad \nabla u + c \ \geq 0, \\ (Au - f) \prod_{i=1}^n \left(\dfrac{\partial u}{\partial x_i} + c_i \right) \ = \ 0 \text{ in } Q, \quad u(\cdot, T) \ = \ 0, \ x \in \Omega. \end{cases} \tag{5.7}$$

We say that u is a strong solution of an evolutionary variational inequality if it satisfy

$$u \in L^2(0, T; H^1(\Omega)), \ \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)), \tag{5.8}$$

$$\begin{gathered} (u_t, v - u) + a(t; u, v - u) \geq (f, v - u) \quad \text{for a.e. } t \in (0, T), \\ \forall v \in H^1(\Omega) \text{ such that } v(x) \geq 0 \text{ a.e in } \Omega, \end{gathered} \tag{5.9}$$

$$u(x, t) \geq 0 \text{ a.e. in } Q, \quad u(\cdot, T) \ = \ 0, \ x \in \Omega. \tag{5.10}$$

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If we introduce the convex set

$$K(t) = \{v \in H^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}, \tag{5.11}$$

then we can reformulate (5.9), (5.10) as follow:

$$\begin{aligned} &u \in K(t), \\ &(u_t, v-u) + a(t; u, v-u) \geq (F, v-u) \quad \text{for a.e. } t \in (0,T) \quad \forall v \in K(t), \text{ a.e. in } t, \end{aligned} \tag{5.12}$$

where

$$(v,u) = \int_\Omega vu dx.$$

Theorem 5.2. *Let the assumptions of Theorem 3.1 be satisfied. Let Ω be the open ball with $\bar{\Omega} \times [0,T] := \bar{Q} \subset \varphi_n$. Then $w \equiv \partial u/\partial x_n + c_n$ is a local solution of* (5.12) *with K given by* (5.11) *and $F \equiv \partial f/\partial x_n + \alpha(t)c_n$.*

Proof. Let $\{\epsilon_k\}_{k=1}^\infty$ be the sequence in Theorem 4.3 and let $u_k = u_{\epsilon k}$. Since $u \in C_{loc}^{2+\mu,1+\mu/2}(\mathbb{R}^n \times [0,T])$ (by Theorem 4.1) then $\dfrac{\partial u}{\partial x_i}$ is continuous for $i = 1, \cdots, n$. Since $\dfrac{\partial u}{\partial x_i} + c_i > 0$ for $i = 1, ..., n-1$ and $\dfrac{\partial u_k}{\partial x_i} \to \dfrac{\partial u}{\partial x_i}$ on $\bar{Q}$ uniformly, then we can assume that $\dfrac{\partial u_k}{\partial x_i} + c_i > 0$ for $i = 1, ..., n-1$ and $k \in \mathbb{N}$ large enough on $\bar{Q}$. Hence $\beta\left(\dfrac{\partial u_k}{\partial x_i} + c_i\right) = 0$ on $\bar{Q}$ for all $k \in \mathbb{N}$, $1 \leq i \leq n-1$. As a result, u_k satisfies

$$Au_k + \frac{1}{\epsilon_k}\beta\left(\frac{\partial u_k}{\partial x_n} + c_n\right) = f, \quad (x,t) \in Q. \tag{5.13}$$

Fix $\delta \in (0,1)$, by Theorem 4.1, $u_k \in C^{2+\delta,1+\delta/2}(\bar{Q})$ and

$$f - \frac{1}{\epsilon_k}\beta\left(\frac{\partial u_k}{\partial x_n} + c_n\right) \in C^{1+\delta,\delta/2}(\bar{Q})$$

then by Theorem 11, page 74 [37] then $u_k \in C^{3+\delta,1+\delta/2}(\bar{Q})$ (that is $D_x^3 u_k, D_x D_t u_k$ exist and are Hölder continuous (exponent δ). Differentiating (5.13) then we have

$$A\frac{\partial u_k}{\partial x_n} + \frac{1}{\epsilon_k}\beta'\left(\frac{\partial u_k}{\partial x_n} + c_n\right)\frac{\partial^2 u_k}{\partial x_n^2} = \frac{\partial f}{\partial x_n}, \quad (x,t) \in Q. \tag{5.14}$$

Defining $w_k \equiv \dfrac{\partial u_k}{\partial x_n} + c_n$ for $k = 1, 2, \cdots$ then for $v \in K(t)$

$$(Aw_k, v - w_k) + \Big(\frac{1}{\epsilon_k}\beta'(w_k)\frac{\partial^2 u_k}{\partial x_n^2}, v - w_k\Big) = \Big(\frac{\partial f}{\partial x_n} + \alpha(t)c_n, v - w_k\Big). \tag{5.15}$$

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We have

$$Aw_k = (w_{kt}, v - w_k) + a(t; w_k, v - w_k)$$

and $\frac{1}{\epsilon_k}\beta'(w_k)\frac{\partial^2 u_k}{\partial x_n^2}.(v-w_k) = 0$ if $w_k \geq v \geq 0$ since $\beta'(y) = 0$ at $y \geq 0$. If $w_k < v$ implies $\beta'(w_k) \leq 0$, but $\frac{\partial^2 u_k}{\partial x_n^2} \geq 0$ implies $\frac{1}{\epsilon_k}\beta'(w_k)\frac{\partial^2 u_k}{\partial x_n^2}.(v - w_k) \leq 0$. Consequently, the second term of (5.15) is non positive. Thus

$$(w_{kt}, (v - w_k)) + a(t; w_k, v - w_k) \geq \Big(\frac{\partial f}{\partial x_n} + \alpha(t)c_n, v - w_k\Big).$$

Since

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(t)\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}dx \leq \liminf_{k\to\infty}\int_\Omega \sum_{i,j=1}^n a_{ij}(t)\frac{\partial w_k}{\partial x_i}\frac{\partial w_k}{\partial x_j}dx.$$

All the other terms of $a(t; w_k, v - w_k)$ converges to their expected limit and $(w_{kt}, (v - w_k))$ also converge to $(w_t, (v - w_k))$. Therefore we have

$$(w_t, (v - w)) + a(t; w, v - w) \geq \Big(\frac{\partial f}{\partial x_n} + \alpha(t)c_n, v - w\Big).$$

Note that $w \geq 0$ a.e in Q because of (5.7). □

Theorem 5.3. *Let the assumptions of Theorem 3.1 hold. Then* $w = \frac{\partial u}{\partial x_n} + c_n \in W^{2,1,\infty}(Q)$ *and* w *satisfy*

$$Aw \geq F, \quad w \geq 0, \quad (Aw - F)w = 0 \text{ a.e. in } Q \tag{5.16}$$

Proof. Let $\bar{Q} = \bar{\Omega} \times [t_0, t_1] \subset Q_B := B \times (t_0', t_1') \subset \bar{Q}_B := \bar{B} \times [t_0', t_1'] \subset \varphi_n$ By Theorem 8.2, [38] $w \in W^{2,1,p}(Q_B)$ for every p with $1 < p < \infty$ implies $w, \frac{\partial w}{\partial x_i}$ are Hölder continuous. Construct $\gamma \in C_0^\infty(Q_B)$ such that $0 \leq \gamma \leq 1$ on Q_B, and $\gamma \equiv 1$ on Q. Compute

$$A(w\gamma) = \gamma Aw - \sum_{i,j=1}^n \Big\{a_{ij}(t)w\frac{\partial^2\gamma}{\partial x_i\partial x_j} + 2a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial \gamma}{\partial x_j}\Big\} - \sum_{i=1}^n g_i(t)\frac{\partial \gamma}{\partial x_i}w - \frac{\partial \gamma}{\partial t}w.$$

Hence $Aw\gamma \geq F^*, w\gamma \geq 0, (Aw\gamma - F^*)w\gamma = 0$, with

$$F^* = \gamma F - \sum_{i,j=1}^n \Big\{a_{ij}(t)w\frac{\partial^2\gamma}{\partial x_i\partial x_j} + 2a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial \gamma}{\partial x_j}\Big\} - \sum_{i=1}^n g_i(t)\frac{\partial \gamma}{\partial x_i}w - \frac{\partial \gamma}{\partial t}w.$$

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Let

$$A_0 u = -\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

We can rewrite

$$A_0(w\gamma) \geq \bar{F}, \quad w\gamma \geq 0$$

and $(Aw\gamma - \bar{F})w\gamma = 0$ a.e. with $\bar{F} = F^* + \sum_{i=1}^{n} g_i(t)\frac{\partial(w\gamma)}{\partial x_i} - \alpha(t)w\gamma$ Since $w, \frac{\partial w}{\partial x_i}$ are Hölder continuous, so is $\bar{F}$. Hence, using Theorem 8.4, [38] implies $w\gamma \in W^{2,1,\infty}_{loc}(Q_B)$ then $w \in W^{2,1,\infty}(Q)$ since $w = w\gamma$ on Q. □

5.1 The Strong Maximum Principle

Theorem 5.4. *Suppose* $\frac{\partial f}{\partial x_i} + \alpha(t)c_i$ *and* $\nabla \frac{\partial f}{\partial x_i}$ *never vanish simultaneously, then* $\frac{\partial f}{\partial x_n} + \alpha(t)c_n < 0$ *on* $\mathcal{F}_n$.

Proof. Since $w \in W^{2,1,\infty}_{loc}(Q)$, implies w and w_{x_i} are continuous. Denote by $\Lambda = \{(x,t) \in Q : w(x,t) = 0\}$ the coincidence set of w and Γ the free boundary, where $\Gamma = \partial\Lambda \cap Q$. If there exist $(x_0, t_0) \in \Gamma$ such that $F(x_0, t_0) > 0$ then $Aw(x_0, t_0) > 0$ and $w(x_0, t_0) = 0$ by Theorem 5.3. Therefore we have in the neighborhood of (x_0, t_0), $Aw(x,t) > 0$ and $w(x,t) \geq 0$. By the strong maximum principle, we have $w(x,t)$ on the neighborhood of (x_0, t_0), implies $(x_0, t_0) \notin \Gamma$. As a result, $F(x,t) \geq 0$ on Γ.

Now we prove that $F(x_0, t_0) > 0$ on Γ. With out loss of generality we assume that (a_{ij}) is the identity matrix since we can make a change of variable if it is necessary. If $F(x_0, t_0) = 0$ for $x_0 \in \Gamma$ then $\nabla F(x_0, t_0) \neq 0$. Suppose that $x_0 = 0$ and that $\frac{\partial F}{\partial x_i} = 0, \quad \forall i = 1, \cdots, n-1$, $\frac{\partial F}{\partial x_n} > 0$ (We can translate and rotate the coordinate if necessary to guarantee our assumption). Since we have $F(x,t) > 0$ if x near x_0 and $x_n > 0$ then there exist R such that $F(x,t) > 0$ in $[K_{\psi^*} \cap B(x_0, R)] \times [t_0, t_1]$. Therefore, $Aw > 0$ in $[K_{\psi^*} \cap B(x_0, R)] \times [t_0, t_1]$, where $K_{\psi^*} = \{x : x_n > 0, cos^{-1}(x_n/|x|) < \psi^*\}$. Since $w \geq 0$ then applying the strong maximum principle, we have $w > 0$ in $[K_{\psi^*} \cap B(x_0, R)] \times [t_0, t_1)$. Fix ϵ, $1 < \lambda < 2$ such that $\lambda + \epsilon < 2$, $r = |x|$. By [89, Lemma 4.7], we

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have

$$\begin{aligned} A\left((t-t_1)^2(r^\lambda f_\lambda(\theta)+r^{\lambda+\epsilon})\right) &< -\frac{\partial}{\partial t}\left((t-t_1)^2(r^\lambda f_\lambda(\theta)+r^{\lambda+\epsilon})\right) \\ &= (t_1-t)r^\lambda(f_\lambda(\theta)+r^\epsilon) \end{aligned} \tag{5.17}$$

Since $\frac{\partial F}{\partial x_i}(0,t_0)=0, \quad \forall i=1,\cdots,n-1$, $\frac{\partial F}{\partial x_n}(0,t_0)>0$ for x near 0 and $x_n>0$ then we have $F(x)\geq ax_n-\delta(|x_1|+\cdots+|x_n|)$ where $\delta\to 0$ as $|x|\to 0$. If $x\in K_{\psi^*}$ then $\frac{x_n}{|x|}>\cos\psi^*$, then for R sufficiently small

$$\begin{aligned} F(x,t) &\geq \frac{a}{2}x_n \\ &\geq \frac{a}{2\cos\psi^*}|x| \\ &= \frac{a}{2\cos\psi^*}r \end{aligned}$$

then $(t_1-t)r^\lambda(f_\lambda(\theta)+r^\epsilon)<F$ for R_1 sufficiently small where $(x,t)\in[K_{\psi^*}\cap B(x_0,R_1)]\times[t_0,t_1]$. Thus

$$\begin{aligned} A\left(w-K(t-t_1)^2(r^\lambda f_\lambda(\theta)+r^{\lambda+\epsilon})\right) &= Aw-KA\left(K(t-t_1)^2(r^\lambda f_\lambda(\theta)+r^{\lambda+\epsilon})\right) \\ &> 0 \quad \text{on} \quad [K_{\psi^*}\cap B(x_0,R_1)]\times[t_0,t_1] \end{aligned}$$

Since $f_\lambda(\psi^*)<0$, there exists $0<R_2\leq R_1$ such that $r^\lambda(f_\lambda(\psi^*)+r^\epsilon)\leq 0, \quad \forall r\leq R_2$ which implies

$$w\geq K(t-t_1)^2r^\lambda(f_\lambda(\psi^*)+r^\epsilon) \quad \text{in} \quad [K_{\psi^*}\cap\partial B(x_0,R_2)]\times[t_0,t_1]$$

Since $w>0$ on $[K_{\psi^*}\cap\partial B(x_0,R_2)]\times[t_0,t_1]$ and V is continuous, then we can assume that $\inf\{w(x,t):(x,t)\in[K_{\psi^*}\cap\partial B(x_0,R_2)]\times[t_0,t_1]\}>0$ we can replace t_1 by a smaller one there exists K such that

$$w\geq K(t-t_1)^2r^\lambda(f_\lambda(\psi^*)+r^\epsilon), \quad \forall(x,t)\in[K_{\psi^*}\cap\partial B(x_0,R_2)]\times[t_0,t_1)$$

At $t=t_1$, we obviously have $w\geq 0$. Hence $w-K(t-t_1)^2r^\lambda(f_\lambda(\theta)+r^\epsilon)\geq 0$ on $[\partial K_{\psi^*}\cap\partial B(x_0,R_2)]\times[t_0,t_1)$ and $[K_{\psi^*}\cap\partial B(x_0,R_2)]\times\{t_1\}$ Using the maximum principle we deduce that

$$w-K(t-t_1)^2r^\lambda(f_\lambda(\theta)+r^\epsilon)\geq 0, \quad \forall(x,t)\in[K_{\psi^*}\cap\partial B(x_0,R_2)]\times[t_0,t_1)$$

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Since $w, \nabla w$ are continuous and $w \equiv 0$ there exists M_1 such that

$$\begin{aligned} w(x,t_0) &\leq M_1|x|^2 \\ &= M_2R^2 \quad \text{in the neighborhood of} \quad x_0 = 0 \end{aligned}$$

However,

$$w(x,t_0) \geq K(t-t_1)^2 r^\lambda (f_\lambda(\theta) + r^\epsilon)$$

implies

$$K(t-t_1)^2 r^\lambda (f_\lambda(\theta) + r^\epsilon) \leq M_2R^2$$

Since $\lambda + \epsilon < 2$ this inequalities do not hold for a small sufficient small r. This contradiction completes the proof. □

5.2 Positive Lebesgue Density For The Coincidence Set

Theorem 5.5. *Let the assumptions as in Theorem 5.4. Then any point $\bar{x} \in \mathcal{F}_n$ is a point of positive Lebesgue density for the coincidence set.*

Proof. Let $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n, \bar{t}) \in \mathcal{F}_n$. Let Ω_0 be an open ball of radius $2R$ with $R > 0$ centered at $\bar{x}$ with $\bar{\Omega}_0 \times [\bar{t}-R, \bar{t}+R] \subset \varphi_n$. By Theorem 5.4 we can take R small enough so that $(\partial f/\partial x_n) + \alpha c_n < 0$ on Ω_0. Since $w(\bar{x}_1, \cdots, \bar{x}_{n-1}, \bar{x}_n + R, t) > 0$, we can take r with $0 < r < R$ so that $w(x,t) > 0$ if $\|x - \bar{x}\| < r, |t - \bar{t}| < r$. Let $\rho = \rho(x,t)$ be the function which assigns to any $x \in \mathbb{R}^n, t \in [0,T]$ its distance to the vertical line through $\bar{x}$, i.e.,

$$\rho(x) = \rho(x_1, \cdots, x_n, t) = [(x_1 - \bar{x}_1)^2 + \cdots + (x_{n-1} - \bar{x}_{n-1})^2 + |t - \bar{t}|.$$

Now define the set

$$D = \{(x_1, \cdots, x_n, t) \in \mathbb{R}^n \times [\bar{t} - r, \bar{t}]; \rho(x_1, \cdots, x_n, t) < r \text{ and } \psi_n(x_1, \cdots, x_{n-1}, t) < x_n < \bar{x}_n + R\}.$$

Since $(\partial f/\partial x_n) + \alpha c_n < 0$ on $\bar{D} \cap \{x_n = \bar{x}_n + R\}$, and the fact that $\partial^2 f/\partial x_n^2 \geq 0$ on $\mathbb{R}^n \times [0,T]$ implies that $(\partial f/\partial x_n) + \alpha c_n < 0$ on $\bar{D}$. Define

$$\eta = \eta(x_1, \cdots, x_n) = \left[\left(\rho(x_1, \cdots, x_n) - \frac{r}{2}\right)^+\right]^4.$$

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Note that $\eta \geq 0$, $\eta \in C^{2,1}(\mathbb{R}^n \times [0,T])$, and that when $\rho \leq r/2$ we have $\eta = 0$. For a large $M > 0$ and for small $\delta > 0$ to be chosen later, for any $\xi = (\xi_1, \cdots, \xi_{n-1}, t) \in \mathbb{R}^{n-1}$ with $|\xi| < \delta$, and for any $x \in \bar{D}$, define

$$W(x,t) \equiv M\frac{\partial w}{\partial x_n}(x,t) + \sum_{k=1}^{n-1} \xi_k \frac{\partial w}{\partial x_k}(x,t) - w(x,k) + (\bar{t} - t)\epsilon\eta(x,t).$$

We will appl the maximum principle to the function $-W$, the operator $-L$, and the set D. Let us make the modification of Theorem 5.1 and our other results which allow Ω to be a cylinder which has been linearly stretched in the x_n-direction. Since the set $\bar{D}$ might be extremely long in the x_n-direction, we will modified such that $\bar{D} \subset \Omega \subset \bar{\Omega} \subset \varphi_n$. This modification is doable becasue if $(\tilde{x}, \tilde{t}) = (\tilde{x}_1, \cdots, \tilde{x}_n, \tilde{t}) \in \mathcal{F}_n$ implies that $(\hat{x}, \tilde{t}) = (\tilde{x}_1, \cdots, \tilde{x}_{n-1}, x_n, \tilde{t}) \in \varphi_n$ for every $x_n < \tilde{\alpha}_n$. To see this claim, assume for contradiction that $\partial u/\partial x_j + c_j = 0$ at $(\hat{x}, \tilde{t}) = (\tilde{x}_1, .., ..\tilde{x}_{n-1}, \hat{x}_n, \tilde{t})$ for some $j = 1, \cdots, n-1$. Define $\Delta = \tilde{x}_n - \hat{x}_n$. Choose $\tau > 0$ such that every point no more that τ units from $\tilde{x}$ in φ_n. let

$$\tilde{x}_n^* = (\tilde{x}_1, \cdots, \tilde{x}_j - \tau, \cdots, \tilde{x}_n) \text{ and } \hat{x}_n^* = (\tilde{x}_1, \cdots, \tilde{x}_j - \tau, \cdots, \tilde{x}_{n-1}, \hat{x}_n).$$

Since $\partial u/\partial x_j \equiv -c_j(\tilde{t})$ on the segment from $\hat{x}_n^*$ to $\hat{x}_n$ and $\partial u/\partial x_n \equiv -c_n(\tilde{t})$ on the segment from $\hat{x}$ to $\tilde{x}$, $u(\tilde{x}, \tilde{t}) - u(\hat{x}^*, \tilde{t}) = -c_j(\tilde{t})\tau - -c_n(\tilde{t})\Delta$. On the other hand, since $\partial u/\partial x_n \geq -c_n(\tilde{t})$ on the segment from $\hat{x}^*$ to $\tilde{x}^*$ and $\partial u/\partial x_j > -c_j(\tilde{t})$ on the segment from $\tilde{x}^*$ to $\tilde{x}$, $u(\tilde{x}) - u(\hat{x}^*) > -c_j(\tilde{t})\tau - -c_n(\tilde{t})\Delta$. This contradiction means that we can assume the aforesaid relationship $\bar{D} \subset \Omega \subset \bar{\Omega} \subset \varphi_n$. By applying the maximum principle, since $Lw \equiv \partial f/\partial x_n + \alpha c_n$ on D,

$$LW = M\frac{\partial^2 f}{\partial x_n^2} + \sum_{k=1}^{n-1} \xi_k \frac{\partial^2 f}{\partial x_k \partial x_n} - \left(\frac{\partial f}{\partial x_n} + \alpha c_n\right) + \epsilon L\eta \text{ on } D.$$

Since $\partial f/\partial x_n < 0$ on $\bar{D}$ while $L\eta$ and all the $\partial^2 f/\partial x_k \partial x_n$ are bounded on $\bar{D}$ and $\partial^2 f/\partial x_n^2 \geq 0$ so it is possible to choose $\epsilon > 0$ and $\delta > 0$ small enough so that $LW \geq 0$ on D whenever $|\xi| < \delta$. Therefore, either $W > 0$ on $\bar{D}$ or W attains its minimum on $\bar{D}$ at some point of ∂D. In view of Theorem 5.3, W is continuous on $\bar{D}$, now we will show $W \geq 0$ on $\bar{D}$. If $W > 0$ then we are done. Hence it suffices to show that $W \geq 0$ on ∂D. More precisely, we will prove $W \geq 0$ on each of the following subset of ∂D^*}.

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i) For the set $\partial D \cap \{w = 0\}$. Let any (x_0, t_0) be in this set, . Since $w(x, t_0) \geq 0$, $w(x, t_0)$ attains maximum at x_0. As a result, $\partial w/\partial x_k = 0$ for $k = 1, \cdot, n$ at this point. Therefore, on this set $W(x, t) \equiv \epsilon\eta(x, t) \geq 0$.

ii) Next, consider $C_\beta = \{(x, t) \in \partial D^*, w(x, t) > 0 \text{ and } \operatorname{dist}(x, \partial D \cap \{w = 0\}) < \beta(\bar{t} - t)\}$, where $\beta > 0$ is chosen small enough so that C_β contain no point (x, t) with $\bar{x}_n + R$. Therefore, for $x \in C_\beta$ we have $\rho(x) = r$. Since $\partial w/\partial x_n = \partial^2 u/\partial x_n^2 \geq 0$, moreover, $w, \partial w/\partial x_1, \cdots, \partial w/\partial x_{n-1}$ are Lipschitz continuous on $\bar{D}$ and are 0 on $\partial D \cap \{w = 0\}$. we can choose $\beta > 0$ small enough such that $W > 0$ on C_β.

iii) Lastly, consider set $E = \{(x, t) \in \partial D; w(x, t) > 0 \text{ and } x \notin C_\beta\}$. At any point $(x, t) \in E, \partial^2 u/\partial x_n^2 > 0$. To see this, assume for contradiction that there is an $(x_0, t_0) \in E$ such that $\partial^2 u/\partial x_n^2(x_0, t_0) = 0$. Since $\partial^2 u/\partial x_n^2 \geq 0$ on $\Big(\Omega \cap \{w > 0\}\Big) \times [\bar{t} - r, \bar{t}]$, then $\partial^2 u/\partial x_n^2$ takes an interior minimum on $\Big(\Omega \cap \{w > 0\}\Big) \times [\bar{t} - r, \bar{t}]$ at (x_0, t_0). By Theorem 5.3, we have $L(\partial^2 u/\partial x_n^2) = \partial^2 f/\partial x_n^2 \geq 0$ on $\Big(\Omega \cap \{w > 0\}\Big) \times [\bar{t} - r, \bar{t}]$ at (x_0, t_0). Using maximum principle then $\partial^2 u/\partial x_n^2 \equiv 0$ on $\Big(\Omega \cap \{w > 0\}\Big) \times [\bar{t} - r, \bar{t}]$, implies $w(x, t_0)$ is constant. This constant must be zero, which gives contradiction. Thus $\partial^2 u/\partial x_n^2 \geq c > 0$ on E for some constant c. Hence there exist some M large enough such that $W \geq 0$ on E. In the summary, we have $W \geq 0$ on ∂D, implies $W \geq 0$ on $\bar{D}$ by the maximum principle. It is because every point in D can be connected to a point in ∂D For $\rho \leq r/2$ we have $\eta \equiv 0$, so on $D \cap \{\rho \leq r/2\}$ we obtain

$$M\frac{\partial w}{\partial x_n}(x, t) + \sum_{k=1}^{n-1} \xi_k \frac{\partial w}{\partial x_k}(x, t) \geq w > 0. \tag{5.18}$$

The left hand sign of (5.18) is the directional derivative of w in the direction $(\xi_1, \cdots, \xi_{n-1}, M, t)$. Let $\bar{\psi}_n(t) = \big(\bar{x}_1, \cdots, \bar{x}_{n-1}, \psi_n(\bar{x}_1, \cdots, \bar{x}_{n-1}, t)\big) \in \mathbb{R}^n$ Consider the region

$$\begin{aligned}\hat{D} = \{(x, t) \in D; \rho(x) < (\bar{t} - t)r/2 \text{ and } (\bar{x} - \bar{\psi}_n(t), t) \text{ is a positive multiple of} \\ (\xi_1, \cdots, \xi_{n-1}, M, t) \text{ for some } \xi \in \mathbb{R}^{n-1} \text{ with } |\xi| < \delta\}\end{aligned} \tag{5.19}$$

If $w(x, t) > 0$ for some $(x, t) \in \hat{D}$, then $(x, t) \in D$. Since the directional derivative of w in $(\xi_1, \cdots, \xi_{n-1}, M)$ (fix t) is positive, when we move from (x, t) to $(\bar{\phi}_n(t), t)$ along this direction,

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$w(x,t)$ is non-decreasing. As a result, $w(\bar{\phi}_n(t),t) > 0$ This contradiction means that $(x,t) \in \hat{D}$ is contained in the coincidence set. it follows that $(\bar{x},\bar{t})$ is a point of positive Lebesgue density for the coincident set.

5.3 Smoothness Of The Free Boundary

Theorem 5.6. *Let the assumptions and notation as in Theorem 5.1. Then in some neighborhood of any point* $(x_0,t_0) \in \mathcal{F}_n$*,*

(i) $\mathcal{F}_n$ *is a* C^1 *hypersurface and, in the* $w > 0$ *region, all second derivatives of* w *are continuous up to* $\mathcal{F}_n$*.*

(ii) If $f \in C^\infty$*, then* $\mathcal{F}_n \in C^\infty$*.*

(iii) If f *is analytic then, for each* T*,* $\mathcal{F}_n \cap \{t = T\}$ *is analytic.*

Proof. i) By applying Theorem 7 of Caffarelli [24] then we have the results. Let $(x_0,t_0) \in \mathcal{F}_n$, according to Theorem 5.5, then (x_0,t_0) is a point of positive Lebesgue density for the coincidence set. Let domain Q contain (x_0,t_0) with $\bar{Q} \subset \varphi_n$. The domain W in Caffarelli's paper is that portion of our Q for which $w > 0$. His v is our w. The solution w satisfy the Stefan problem (5.11), (5.12) by Theorem 5.2. Our problem satisfied the condition (PH1),(PH2), (PH3) in the paper of Caffarelli. The (PH1) condition about Q is known to be increasing in time is proved by following the proof of Theorem 9.1 in Friedman [38], for any $t \geq 0$, consider

$$N(t) = \{x \in \Omega; w(x,t) > 0\}$$

They proved that $N(t) \subset N(t')$ if $t < t'$, which implied Q is known to be increasing in time. The (PH2) condition is satisfied from Theorem 5.3 we have $w \in W^{1,2,\infty}(Q)$ so that $w \in C^{1,1}(Q)$. Moreover, the solution w satisfies $0 \leq w_t \leq C, \quad C < \infty$ (see the proof in Friedman [38], Theorem 9.1, page 84]). The last hypothesis we have to check (PH3) is satisfied since $w \in C^{1,1}(Q)$, so $w(x_0,t_0) = 0$ and $\nabla w(x_0,t_0) = 0$. Moreover, we have $w \geq 0$. The $\partial_1 W$ of [24] is our $\varphi_n \cap Q$. As we mentioned above, w and ∇w are zero on this set. ii) The assertion of our theorem follow from Theorem 3 of Kinderlehrer and Nirenberg [55]. Their u is our w, there Ω is our $\{(x,t) \in$

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$\Omega : w(x,t) > 0\}$, their general nonlinear parabolic equation $u_t - F(x,t,u,D_x u, D_x^2 u) = 0$ is our $Aw - \partial f/\partial x_n - \alpha c_n = 0$. Their boundary condition are $u = |\nabla u| = 0$ on Γ satisfied since our w and ∇w are zero on $\mathcal{F}_n$ implies w and $|\nabla w|$ both are equal zero on $\mathcal{F}_n$. We have $(x_0, t_0) \in \mathcal{F}_n$, the condition $F(0, \cdots, 0) \neq 0$ become $\partial f/\partial x_n - \alpha c_n \neq 0$ at (x_0, t_0), which was proved in Theorem 5.4. Condition (I) of Kinderlehrer and Nirenberg [55] requires that the boundary of Ω is a $C^{1,1}$ hypersurface Γ-the free boundary. (II)': u and u_{x^i} belong to $C^{1,1}$ in $\Omega \cup \Gamma, i = 1, \cdots, n$. That assures by i). iii) According to Theorem 3' of [55], if we also assumed that f is real analytic then , for each T, $\mathcal{F}_n \cap \{t = T\}$ is analytic. □

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CHAPTER 6 REGULARIZATION FOR A NONLINEAR BACKWARD PARABOLIC PROBLEM WITH CONTINUOUS SPECTRUM OPERATOR

6.1 Regularization of the homogeneous problem

In this section, we shall consider the homogeneous problem

$$u_t + Au(t) = 0, \ 0 < t < T, \tag{6.1}$$

$$u(T) = \varphi. \tag{6.2}$$

To proceed, we present the notation and the functional setting which will be use in the two last sections. Let a be a positive number. We denote by $\{E_\lambda, \ \lambda \geq a\}$ the spectral resolution of the identity associated to A.

We denote by $S(t) = e^{-tA} = \int_a^\infty e^{-t\lambda}\, dE_\lambda \in \mathcal{L}(H)$, $t \geq 0$, the C_0-semi-group generated by $-A$. Some basic properties of $S(t)$ are listed in the following theorem.

Theorem 6.1. *(see [32], Ch.2, Theorem 6.13, p.74). For the family of operators $S(t)$, the following properties are valid:*

1. $\|S(t)\| \leq 1$, *for all* $t \geq 0$;
2. *the function* $t \longmapsto S(t)$, $t > 0$, *is analytic;*
3. *for every real* $r \geq 0$ *and* $t > 0$, *the operator* $S(t) \in \mathcal{L}(H, \mathcal{D}(A^r))$;
4. *for every integer* $k \geq 0$ *and* $t > 0$, $\|S^{(k)}(t)\| = \|A^k S(t)\| \leq c(k)t^{-k}$;
5. *for every* $x \in \mathcal{D}(A^r)$, $r \geq 0$ *we have* $S(t)A^r x = A^r S(t)x$.

Definition 6.3. *Let $A : D(A) \subset H \longrightarrow H$ be a self-adjoint operator on the Hilbert space H over K and let $f, g : \mathbb{R} \longrightarrow K$ be piecewise continuous function. We set*

$$D(f(A)) = \{u \in H : \int_a^{+\infty} |f(\lambda)|^2 d\|E_\lambda u\|^2 < \infty\}$$

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and define the linear operator $f(A): D(A) \subset H \longrightarrow H$ *by the formula*

$$f(A)u = \int_a^{+\infty} f(\lambda) dE_\lambda u,$$

for all $u \in D(f(A))$.

It is useful to know exactly the admissible set for which (6.1)-(6.2) has a solution. The following lemma gives an answer to this question.

Lemma 6.1. *Problem* (6.1) – (6.2) *has a solution if and only if*

$$\int_a^{\infty} e^{2\lambda T} d\|E_\lambda \varphi\|^2 < \infty$$

and its unique solution is represented by

$$u(t) = e^{(T-t)A}\varphi. \tag{6.3}$$

If the problem (6.1) – (6.2) admits a solution u then this solution can be represented by

$$u(t) \quad = \quad e^{(T-t)A}\varphi = \int_a^{\infty} e^{\lambda(T-t)} dE_\lambda \varphi. \tag{6.4}$$

Since $t < T$, we know from (6.4) that the terms $e^{-(t-T)\lambda}$ is the source of the instability. So, to regularize problem (6.4),we should replace it by the better terms. Let φ and φ_ϵ denote the exact and measured data at $t = T$, respectively, which satisfy

$$\|\varphi - \varphi_\epsilon\| \leq \epsilon.$$

In this section, we perturbed the final condition $u(T) = \varphi$ to form an approximate nonlocal problem depending on a small parameter. We introduced the regularized problem with boundary condition containing a derivative of the same order than the equation as the following equation

$$v_t^\epsilon + Av^\epsilon = 0, \;\; 0 < t < T + m \tag{6.5}$$

$$\epsilon v_t^\epsilon(0) + v^\epsilon(T+m) = \varphi_\epsilon \tag{6.6}$$

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where $m > 0$ is a fixed number.

It is standard to prove the well-posedness of (6.5)–(6.6) and the representation of its solution

$$u^\epsilon(t) = \int_a^\infty \frac{e^{-\lambda t}}{\epsilon\lambda + e^{-\lambda(T+m)}} dE_\lambda \varphi,\ \ t \in [0, T+m]. \tag{6.7}$$

To get an error estimate for $\|v^\epsilon(t+m) - u(t)\|$, we will use the function

$$v^\epsilon(t) = \int_a^\infty \frac{e^{-\lambda t}}{\epsilon\lambda + e^{-\lambda(T+m)}} dE_\lambda \varphi_\epsilon,\ \ t \in [0, T+m]. \tag{6.8}$$

Theorem 6.2. *Assume that u has the eigenfunction expansion $u(t) = \int_0^\infty dE_\lambda u(t)$.*

a) Assume that there exist a positive constant C_1 such that $\|Au(0)\| \le C_1$. Then for every $t \in [0, T]$,

$$\|v^\epsilon(t+m) - u(t)\| \le \epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}} + \frac{C_1}{\ln(\frac{T+m}{\epsilon})}. \tag{6.9}$$

b) Assume that there exist some positive constants m and C_2 such that

$$\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \le C_2^2. \tag{6.10}$$

Then for every $t \in [0, T]$,

$$\|v^\epsilon(t+m) - u(t)\| \le (C_2+1)\epsilon^{\frac{t+m}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}}. \tag{6.11}$$

First, we considered the Lemma which is useful to this paper.

Lemma 6.1. *Let $s, t, \epsilon, m, \xi, \lambda$ be real numbers such that $0 \le t \le s \le T$, $\lambda \in (0, \infty)$ and $\epsilon > 0$. Then the following estimate holds true*

$$\frac{e^{-(t+m)\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}} \le \epsilon^{\frac{t-T}{T+m}} \left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}}. \tag{6.12}$$

Proof. By using the inequality

$$\frac{1}{\epsilon\lambda + e^{-(T+m)\lambda}} \le \frac{1}{\epsilon \ln(\frac{T+m}{\epsilon})} \tag{6.13}$$

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we obtain

$$\begin{aligned}\frac{e^{-(t+m)\lambda}}{\epsilon\lambda+e^{-(T+m)\lambda}} &= \frac{e^{-(t+m)\lambda}}{\left(\epsilon\lambda+e^{-(T+m)\lambda}\right)^{\frac{t+m}{T+m}}\left(\epsilon+e^{-(T+m)\lambda}\right)^{\frac{T-t}{T+m}}} \\ &\leq \frac{1}{\left(\epsilon\lambda+e^{-(T+m)\lambda}\right)^{\frac{T-t}{T+m}}} \\ &\leq \left(\frac{T+m}{\epsilon\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}} \\ &= \epsilon^{\frac{t-T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}}. \end{aligned} \tag{6.14}$$

Step 1. Estimate $\|v^\epsilon(t+m)-u^\epsilon(t+m)\|$. For every $\lambda\in(0,\infty)$,

$$v^\epsilon(t+m)-u^\epsilon(t+m)=\int_a^\infty \frac{e^{-\lambda(t+m)}}{\epsilon\lambda+e^{-\lambda(T+m)}}dE_\lambda(\varphi_\epsilon-\varphi)$$

we have

$$\begin{aligned}\|v^\epsilon(t+m)-u^\epsilon(t+m)\|^2 &= \int_a^\infty\left(\frac{e^{-\lambda(t+m)}}{\epsilon\lambda+e^{-\lambda(T+m)}}\right)^2 d\|E_\lambda(\varphi_\epsilon-\varphi)\|^2 \\ &\leq \epsilon^{\frac{2t-2T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}\int_a^\infty d\|E_\lambda(\varphi_\epsilon-\varphi)\|^2 \\ &\leq \epsilon^{\frac{2t-2T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}\|(\varphi_\epsilon-\varphi)\|^2 \\ &\leq \epsilon^{\frac{2t-2T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}\epsilon^2 \\ &\leq \epsilon^{\frac{2t+2m}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}. \end{aligned}$$

Thus

$$\|v^\epsilon(t+m)-u^\epsilon(t+m)\|\leq \epsilon^{\frac{t+m}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}}. \tag{6.15}$$

Step 2. We estimate $\|u^\epsilon(t+m)-u(t)\|$ if $\|Au(0)\|\leq C_1$.

$$\begin{aligned}u^\epsilon(t+m)-u(t) &= \int_a^\infty\left(\frac{e^{-\lambda(t+m)}}{\epsilon\lambda+e^{-\lambda(T+m)}}-e^{(T-t)\lambda}\right)dE_\lambda\varphi \\ &= \int_a^\infty \frac{\epsilon\lambda e^{(T-t)\lambda}}{\epsilon\lambda+e^{-(T+m)\lambda}}dE_\lambda\varphi \\ &= \int_a^\infty \frac{\epsilon\lambda}{\epsilon\lambda+e^{-(T+m)\lambda}}dE_\lambda u(0). \end{aligned} \tag{6.16}$$

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And applying the inequality (6.13) again, we obtain

$$\begin{aligned}\|u^\epsilon(t+m)-u(t)\|^2 &= \int_a^\infty \Big(\frac{\epsilon\lambda}{\epsilon\lambda+e^{-(T+m)\lambda}}\Big)^2 d\|E_\lambda u(0)\|^2 \\ &\le \Big(\frac{\epsilon}{\epsilon\ln(\frac{T+m}{\epsilon})}\Big)^2 \int_a^\infty \lambda^2 d\|E_\lambda u(0)\|^2 \\ &\le \Big(\frac{1}{\ln(\frac{T+m}{\epsilon})}\Big)^2 \|Au(0)\|^2.\end{aligned}$$

Therefore

$$\|u^\epsilon(t+m)-u(t)\| \le \frac{1}{\ln(\frac{T+m}{\epsilon})}\|Au(0)\|.$$

Applying the triangle inequality, we get

$$\begin{aligned}\|v^\epsilon(t+m)-u(t)\| &\le \|v^\epsilon(t+m)-u^\epsilon(t+m)\| + \|u^\epsilon(t+m)-u(t)\| \\ &\le \epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}} + \frac{1}{\ln(\frac{T+m}{\epsilon})}\|Au(0)\|.\end{aligned}$$

Step 3. Estimate $\|u^\epsilon(t+m)-u(t)\|$ if $\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \le C_2^2$.

Since $u^\epsilon(t+m)-u(t) = \int_a^\infty \frac{\epsilon\lambda}{\epsilon\lambda+e^{-(T+m)\lambda}} dE_\lambda u(0)$, we would get

$$\begin{aligned}u^\epsilon(t+m)-u(t) &= \int_a^\infty \frac{\epsilon\lambda}{\epsilon\lambda+e^{-(T+m)\lambda}} dE_\lambda u(0) \\ &= \int_a^\infty \frac{\epsilon\lambda e^{-(t+m)\lambda}}{\epsilon\lambda+e^{-(T+m)\lambda}} e^{(t+m)\lambda} dE_\lambda u(0).\end{aligned}$$

Then

$$\begin{aligned}\|u^\epsilon(t+m)-u(t)\|^2 &= \int_a^\infty \Big(\frac{\epsilon e^{-(t+m)\lambda}}{\epsilon\lambda+e^{-(T+m)\lambda}}\Big)^2 \Big(\lambda e^{(t+m)\lambda}\Big)^2 d\|E_\lambda u(0)\|^2 \\ &\le \epsilon^2 \epsilon^{\frac{2t-2T}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{2T-2t}{T+m}} \int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \\ &= \epsilon^{\frac{2t+2m}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{2T-2t}{T+m}} \int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2.\end{aligned}$$

Thus

$$\|u^\epsilon(t+m)-u(t)\| \le \epsilon^{\frac{t+m}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}} \sqrt{\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2}.$$

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Applying the triangle inequality, we obain

$$\begin{aligned}\|v^\epsilon(t+m)-u(t)\| &\leq \|v^\epsilon(t+m)-u^\epsilon(t+m)\|+\|u^\epsilon(t+m)-u(t)\|\\ &\leq \epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}\\ &+ \epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}\sqrt{\int_a^\infty \lambda^2 e^{2(t+m)\lambda}d\|E_\lambda u(0)\|^2}\\ &\leq (C_2+1)\epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}.\end{aligned}$$

□

6.2 Regularization of the nonlinear problem

In this section, we shall approximate the problem (1.9) – (1.10) by the following problem as follows

$$\frac{d}{dt}u^\epsilon(t)+A_\epsilon u^\epsilon(t)=B(\epsilon,t)f(t,u^\epsilon(t)),\quad t\in(0,T), \tag{6.17}$$

$$u^\epsilon(T)=\varphi, \tag{6.18}$$

where $A_\epsilon, B(\epsilon,t)$ are defined in (6.19) and (6.20). For every $v\in H$ having the expansion $v=\int_a^{+\infty}dE_\lambda v$, we define

$$S(t)v=\int_a^{+\infty}e^{-t\lambda}dE_\lambda v.$$

$$A_\epsilon(v)=-\frac{1}{T+m}\int_a^{+\infty}\ln(\epsilon+e^{-(T+m)\lambda})dE_\lambda v. \tag{6.19}$$

$$B(\epsilon,t)(v)=\int_a^{+\infty}(1+\epsilon e^{(T+m)\lambda})^{\frac{t-T}{T+m}}dE_\lambda v,\ t\in[0,T]. \tag{6.20}$$

$$(\epsilon I+S(T+m))^{\frac{t-T}{T+m}}v=\int_a^{+\infty}\frac{dE_\lambda v}{(\epsilon+e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}} \tag{6.21}$$

Notice that if $f=0$ then the problem (6.17) – (6.18) has been studied in [20]. The main theorem is as follows

Theorem 6.1. *Let $\varphi\in H$. Then the problem* (6.17) – (6.18) *has a unique solution $u^\epsilon\in H$. Let $m>0$ be a positive number. Let $u\in C([0,T];H)$ be a solution of* (6.1)–(6.2)*. Assume that u*

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has the eigenfunction expansion $u(t) = \int_a^{+\infty} dE_\lambda u(t)$ *satisfying* $\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_\lambda u(t)\|^2 < \infty$ *for every* $t \in (0,T]$. *Let* φ_ϵ *be a measured data such that* $\|\varphi_\epsilon - \varphi\| \le \epsilon$ *where* $\epsilon \in (0, min\{T, 1-e^{-Ta}\})$. *Using* φ_ϵ, *we can construct a function* $U^\epsilon : [0,T] \longrightarrow H$ *such that*

$$\|U^\epsilon(t) - u(t)\| \le (1+B)e^{k(T-t)}\epsilon^{\frac{t+m}{T+m}}, \ \forall t \in (0,T],$$

where $\epsilon \in (0, min\{T, 1-e^{-Ta}\})$, U^ϵ *is a solution of* (6.17) *with* $U^\epsilon(T) = \varphi_\epsilon$ *and*

$$B = \sup_{t\in[0,T]} \sqrt{\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_\lambda u(t)\|^2}.$$

First, we introduce some useful lemmas of the several results in this dissertation.

Lemma 6.2. *Let* $\epsilon > 0$ *and* $0 < t < s < T$. *Let* A_ϵ *be defined in* (6.19) *where* $\epsilon \in (0, 1-e^{-Ta})$. *Let* $B(\epsilon, t)$ *be defined in* (6.20). *Then the following inequalities hold:*

a) $\|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \le \epsilon^{\frac{t-T}{T+m}}$

b) $\|S(T-s)(\epsilon I + S(T+m))^{\frac{t}{T}-1}\| \le \epsilon^{\frac{t-s}{T+m}}$.

c) $\|A_\epsilon\| \le \dfrac{1}{T}\ln(\dfrac{1}{\epsilon})$.

d) $\|B(\epsilon,t)\| \le 1$.

Proof. *a.* Let $v \in H$ and let $v = \int_a^{+\infty} dE_\lambda v$ be the eigenfunction expansion of v, we have

$$\begin{aligned} \|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} v\|^2 &= \int_a^{+\infty} \frac{d\|E_\lambda v\|^2}{(\epsilon + e^{-(T+m)\lambda})^{\frac{2T-2t}{T+m}}} \\ &\le \int_a^{+\infty} \frac{d\|E_\lambda v\|^2}{\epsilon^{\frac{2T-2t}{T+m}}} = \epsilon^{\frac{2t-2T}{T+m}}\|v\|^2. \end{aligned}$$

Therefore, we obtain

$$\|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \le \epsilon^{\frac{t-T}{T+m}}.$$

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b. First, letting $v \in H$, we get

$$\begin{aligned}
& \|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(v)\|^2 \\
= & \int_a^{+\infty} e^{2(s-T)\lambda}(\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2T}{T+m}} d\|E_\lambda v\|^2 \\
= & \int_a^{+\infty} (\epsilon e^{(T+m)\lambda}+1)^{\frac{2s-2T}{T+m}}(\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
\leq & \int_a^{+\infty} (\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
\leq & \int_a^{+\infty} \epsilon^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
= & \epsilon^{\frac{2t-2s}{T+m}}\|v\|^2.
\end{aligned}$$

Then, the following inequality is obtained

$$\|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \leq \epsilon^{\frac{t-s}{T+m}}.$$

c. We have

$$\|A_\epsilon(v)\|^2 = \frac{1}{(T+m)^2}\int_a^{+\infty} \ln^2(\frac{1}{\epsilon+e^{-(T+m)\lambda}})d\|E_\lambda v\|^2.$$

Since $\epsilon \in (0, 1-e^{-Ta})$, we obtain $\epsilon + e^{-(T+m)\lambda} < 1,\ \forall\lambda \geq a$. The next following step is that

$$0 < \ln(\frac{1}{\epsilon+e^{-(T+m)\lambda}}) < \ln(\frac{1}{\epsilon}).$$

For that reason, we will get

$$\|A_\epsilon(v)\|^2 \leq \frac{1}{(T+m)^2}\ln^2(\frac{1}{\epsilon})\int_a^{+\infty} d\|E_\lambda v\|^2 \leq \frac{1}{T^2}\ln^2(\frac{1}{\epsilon})\|v\|^2.$$

d. Taking $v \in H$, we have

$$\|B(\epsilon,t)(v)\|^2 = \int_a^{+\infty}(1+\epsilon e^{(T+m)\lambda})^{\frac{2t-2T}{T+m}} d\|E_\lambda v\|^2 \leq \int_a^{+\infty} d\|E_\lambda v\|^2 = \|v\|^2,$$

which concludes the proof. □

Lemma 6.3. *Let $\varphi \in H$ and let $f : \mathbb{R}\times H \longrightarrow H$ be a continuous operator satisfying $\|f(t,w) - f(t,v)\| \leq k\|w-v\|$ for a $k>0$ independent of $w, v \in H, t \in \mathbb{R}$. Then problem (6.17)–(6.18) has a unique solution $u^\epsilon \in C([0,T];H)$ for any $0<\epsilon<1-e^{-T\lambda_1}$.*

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Proof. The proof of Lemma 6.3 is divided into three steps.

Step 1.

For $w \in C([0,T];H)$, we insert

$$F(w)(t) = (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\Big[\varphi - \int_t^T S(T-s)f(s,w(s))ds\Big]. \tag{6.22}$$

For every $w, v \in C([0,T];H)$, we stated that we will have

$$\|F^n(w)(t) - F^n(v)(t)\| \leq \left(\frac{k(T-t)}{\epsilon}\right)^n \frac{C^n}{n!}|||w-v|||, \tag{6.23}$$

where $C = \max\{T, 1\}$ and $|||.|||$ is sup norm in $C([0,T];H)$. Using the induction method we shall prove the latter inequality. Using Lemma 6.2 and the Lipschitz property of f for $n=1$, we have

$$\begin{aligned} \|F(w)(t) - F(v)(t)\| &= \|\int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(f(s,w(s)) - f(s,v(s))ds\| \\ &\leq \frac{k}{\epsilon}\int_t^T \|w(s)-v(s)\|ds \leq C\frac{k}{\epsilon}(T-t)|||w-v|||. \end{aligned}$$

Suppose that (6.23) holds for $n = j$. We would prove that (6.23) holds for $n = j+1$. In fact, we have

$$\begin{aligned} \|F^{j+1}(w)(t) - F^{j+1}(v)(t)\| &= \|\int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(f(F^j w)(s) - f(F^j v)(s))ds\| \\ &\leq \frac{1}{\epsilon}(T-t)k\int_t^T \|F^j(w)(s) - F^j(v)(s)\|^2 ds \\ &\leq \left(\frac{k}{\epsilon}\right)^{(j+1)} \frac{(T-t)^{j+1}}{(j+1)!} C^{j+1}|||w-v|||. \end{aligned}$$

Therefore, by the induction principle, we have (6.23) for all $w, v \in C([0,T];H)$. We consider $F : C([0,T];H) \longrightarrow C([0,T];H)$. Since $\lim\limits_{n\longrightarrow\infty}\left(\frac{kT}{\epsilon}\right)^n\frac{C^n}{n!} = 0$, there exists a positive integer n_0 such that F^{n_0} is a contraction. It follows that the equation $F^{n_0}(w) = w$ has a unique solution $u^\epsilon \in C([0,T];H)$.

We claim that $F(u^\epsilon) = u^\epsilon$. In fact, one has $F(F^{n_0}(u^\epsilon)) = F(u^\epsilon)$. Hence $F^{n_0}(F(u^\epsilon)) = F(u^\epsilon)$. By the uniqueness of the fixed point of F^{n_0}, one has $F(u^\epsilon) = u^\epsilon$, i.e., the equation $F(w) = w$ has a unique solution $u^\epsilon \in C([0,T];H)$.

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Step 2.

Suppose u^ϵ is the unique solution of the integral equation (6.22), then u^ϵ is also a solution of the (6.17)–(6.18).

In fact, we have

$$u^\epsilon(t) = (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\Big[\varphi - \int_t^T S(T-s)f(s,u^\epsilon(s))ds\Big],\ t \in [0,T]. \tag{6.24}$$

Taking the derivative of $u^\epsilon(t)$, and by direct computation, we get

$$\frac{d}{dt}u^\epsilon(t) = A_\epsilon u^\epsilon(t) + B(\epsilon,t)f(t,u^\epsilon(t)). \tag{6.25}$$

Now, we are clear to see that

$$u^\epsilon(T) = \varphi.$$

Hence, u^ϵ is a solution of problem (6.17)–(6.18).

Step 3. The problem (6.17)–(6.18) has at most one solution in $C([0,T];H)$.

Let u and v be two solutions of problem (6.17)–(6.18) such that $u, v \in C([0,T];H)$. First, we denote $g : \mathbb{R} \times H \longrightarrow H$ such that

$$g(t,u(t)) = B(\epsilon,t)f(t,u(t)).$$

Next, because the property of function f defined by Theorem 4.3, for any $w, v \in H$, we have for any $w, v \in H$

$$\|g(t,w(t)) - g(t,v(t))\| \le \|B(\epsilon,t)\|\|f(t,w) - f(t,v)\| \le k\|w - v\|.$$

Put

$$w(t) = e^{-b(t-T)}(u(t) - v(t))\ b > 0.$$

By calculating directly, we have w satisfying the equation

$$w_t + A_\epsilon w(t) - bw(t) = e^{b(t-T)}\left(g(t,e^{-b(t-T)}u(t)) - g(t,e^{-b(t-T)}v(t))\right). \tag{6.26}$$

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It follows that

$$< w_t(t) + A_\epsilon w(t) - bw(t), w(t) > = < e^{b(t-T)} \left(g(t, e^{-b(t-T)}u(t)) - g(t, e^{-b(t-T)}v(t)) \right), w(t) > .$$

Using the Lipschitz property of f, we have

$$| < e^{b(t-T)} \left(g(t, e^{-b(t-T)}u(t)) - g(t, e^{-b(t-T)}v(t)) \right), w(t) > | \ \leq \ k\|w(t)\|^2,$$

the result is

$$< e^{b(t-T)} \left(g(t, e^{-b(t-T)}u(t)) - g(t, e^{-b(t-T)}v(t)) \right), w(t) > \geq \ -k\|w(t)\|^2$$

and using Lemma 6.2c, we have

$$| < A_\epsilon w(t), w(t) > | \ \leq \frac{1}{T} \ln(\frac{1}{\epsilon})\|w(t)\|^2,$$

which gives

$$< A_\epsilon w(t), w(t) > \geq \ -\frac{1}{T} \ln(\frac{1}{\epsilon})\|w(t)\|^2.$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 \ \geq b\|w(t)\|^2 - k\|w(t)\|^2 - \frac{1}{T} \ln(\frac{1}{\epsilon})\|w(t)\|^2.$$

Let any $t_1 \in [0, T]$. Taking the integral with respect to t from t_1 to T, we get

$$\|w(T)\|^2 - \|w(t_1)\|^2 \geq 2 \int_{t_1}^{T} (b - k - \frac{1}{T} \ln(\frac{1}{\epsilon}))\|w(t)\|^2 dt.$$

Choosing $b = k + \frac{1}{T}\ln(\frac{1}{\epsilon})$ and noting that $w(T) = 0$, we get $w(t_1) = 0$. Hence, $w(t) = 0$ or $u(t) = v(t), \ \forall t \in [0, T]$. This has concluded the proof of step 3. □

Lemma 6.4. *The (unique) solution of problem* (6.17)–(6.18) *depends continuously (in* $C([0, T]; H)$*) on* φ.

Proof. Let u and v be two solutions of problem (6.17)–(6.18) corresponding to the final values φ and ω respectively. We have

$$\begin{aligned} u(t) - v(t) \ &= \ (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(\varphi - \omega) \\ &- \int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(f(u(s) - f(v(s))ds. \end{aligned}$$

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When we applied Lemma 6.2 and the Lipchitz property of f we get

$$\begin{aligned} \|u(t)-v(t)\| &\leq \|(\epsilon I+S(T+m))^{\frac{t-T}{T+m}}(\varphi-\omega)\| \\ &\quad +\|\int_t^T S(T-s)(\epsilon I+S(T+m))^{\frac{t-T}{T+m}}(f(u(s))-f(v(s))ds\| \\ &\leq \epsilon^{\frac{t-T}{T+m}}\|\varphi-\omega\|+k\int_t^T \epsilon^{\frac{t-s}{T+m}}\|u(s)-v(s)\|ds. \end{aligned}$$

Therefore

$$\epsilon^{\frac{-t}{T+m}}\|u(t)-v(t)\| \leq \epsilon^{-\frac{T}{T+m}}\|\varphi-\omega\|+k\int_t^T \epsilon^{-\frac{s}{T+m}}\|u(s)-v(s)\|ds.$$

Applying Gronwall's inequality, we obtain

$$\|u(t)-v(t)\| \leq \epsilon^{\frac{t-T}{T+m}}e^{k(T-t)}\|\varphi-\omega\|.$$

The solution of the problem (6.17)–(6.18) is depending continuously on φ and Lemma 6.4 proof using the inequality as stated. □

Now, we turn to

Proof of Theorem 6.1. In view of (1.11) that

$$u(t)=S(t-T)\varphi-\int_t^T S(t-s)f(u(s))ds.$$

It leads to

$$\begin{aligned} S(T-t)(\epsilon I+S(T+m))^{(t-T)/T+m}u(t) &= (\epsilon I+S(T+m))^{\frac{t-T}{T+m}}\varphi- \\ &\quad -\int_t^T S(T-s))(\epsilon I+S(T+m))^{\frac{t-T}{T+m}}f(u(s))ds. \end{aligned}$$

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Applying Lemma 6.2 and the inequality $1-(1+x)^{-\alpha} \leq x\alpha,\ (x, \alpha > 0)$ we get

$$\begin{aligned}
\|u(t)-u^{\epsilon}(t)\| &\leq \int_t^T \|S(T-s)(\epsilon I+S(T+m))^{\frac{t-T}{T+m}}\|\ \|f(u(s))-f(u^{\epsilon}(s))\|ds \\
&+ \|(I-S(T-t)(\epsilon I+S(T+m))^{\frac{t-T}{T+m}})u(t)\| \\
&\leq k\int_t^T \epsilon^{\frac{t-s}{T+m}}\|u(s)-u^{\epsilon}(s)\|ds \\
&+ \sqrt{\int_a^{+\infty}\left(1-(1+\epsilon e^{(T+m)\lambda}\right)^{\frac{t-T}{T+m}})^2 d\|E_\lambda u(t)\|^2} \\
&\leq k\epsilon^{\frac{t+m}{T+m}}\int_t^T \epsilon^{-\frac{s+m}{T+m}}\|u(s)-u^{\epsilon}(s)\|ds+\epsilon\frac{T-t}{T+m}\sqrt{\int_a^{+\infty} e^{2(T+m)\lambda}d\|E_\lambda u(t)\|^2}.
\end{aligned}$$

Notice that $0<\epsilon<\epsilon^{\frac{t+m}{T+m}}$ we obtain

$$\epsilon^{\frac{-t-m}{T+m}}\|u(t)-u^{\epsilon}(t)\| \leq B+k\int_t^T \epsilon^{\frac{-s-m}{T+m}}\|u(s)-u^{\epsilon}(s)\|ds.$$

By using Gronwall's inequality we get $\epsilon^{\frac{-t-m}{T+m}}\|u(t)-u^{\epsilon}(t)\| \leq Be^{k(T-t)}$.

Therefore

$$\|u(t)-u^{\epsilon}(t)\| \leq Be^{k(T-t)}\epsilon^{\frac{t+m}{T+m}}.$$

This follows from Lemma 4.3, Lemma 6.4 that

$$\begin{aligned}
\|U^{\epsilon}(t)-u(t)\| &\leq \|w^{\epsilon}(t)-u^{\epsilon}(t)\|+\|u^{\epsilon}(t)-u(t)\| \\
&\leq (1+B)e^{k(T-t)}\epsilon^{\frac{t+m}{T+m}},
\end{aligned}$$

for every $t \in (0, T)$.

This has concluded the proof of Theorem 6.1. □

6.3 Numerical experiment

In this section we give an explicit example for problem (1.9) – (1.10). Let us consider the backward heat problem

$$-u_{xx}+u_t = f(u)+g(x,t), \quad (x,t) \in (0,\pi)\times(0,1) \tag{6.27}$$

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$$u(0,t) = u(\pi,t) = 0, \quad t \in [0,1], \tag{6.28}$$

$$u^{\epsilon}(x,1) = \varphi(x), \;\; x \in [0,\pi] \tag{6.29}$$

where

$$g(x,t) = 2e^t \sin x - e^{4t}\sin^4 x,$$

$$f(u) = \begin{cases} u^4 & u \in [-e^{10}, e^{10}] \\ -\frac{e^{10}}{e-1}u + \frac{e^{41}}{e-1} & u \in (e^{10}, e^{11}] \\ \frac{e^{10}}{e-1}u + \frac{e^{41}}{e-1} & u \in (-e^{11}, -e^{10}] \\ 0 & |u| > e^{11} \end{cases}$$

and

$$u(x,1) = \varphi_0(x) \equiv e \sin x.$$

This is a particular case of (1.9) – (1.10) where $H = L^2(0,\pi)$ and $A = -\Delta$, which associates with the homogeneous Dirichlet boundary condition. This operator admits an eigenbasis $\phi_x = \frac{2}{\pi}\sin(nx)$ for $L^2(0,\pi)$ corresponding to the eigenvalues $\lambda_n = n^2$. We also denote $< . >$ is the inner product in $L^2(0,\pi)$. The exact solution of the equation is

$$u(x,t) = e^t \sin x.$$

Especially

$$u\left(x, \frac{999}{1000}\right) \equiv u(x) = exp\left(\frac{999}{1000}\right)\sin x \approx 2.715564905 \sin x.$$

Let $\varphi_\epsilon(x) \equiv \varphi(x) = (\epsilon+1)e\sin x$. We have

$$\|\varphi_\epsilon - \varphi\|_2 = \sqrt{\int_0^\pi \epsilon^2 e^2 \sin^2 x dx} = \epsilon e \sqrt{\frac{\pi}{2}}.$$

From (6.19) and (6.20) we have

$$A_\epsilon(v) = -\frac{1}{T}\sum_{n=1}^{\infty} \ln(\epsilon + e^{-Tn^2}) v_n \sin nx$$

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and

$$B(\epsilon,t)(v)=\sum_{n=1}^{\infty}(1+\epsilon e^{Tn^2})^{\frac{t}{T}-1}v_n\sin nx$$

for $v=\sum\limits_{n=1}^{\infty}v_n\sin nx$.

Applying the problem (6.17) – (6.18), we have the regularized problem

$$\frac{\partial}{\partial t}u^\epsilon(x,t)-\sum_{n=1}^{\infty}\ln(\epsilon+e^{-n^2})u_n^\epsilon\sin nx=\sum_{n=1}^{\infty}(1+\epsilon e^{Tn^2})^{t-1}f_n(u_\epsilon)\sin nx,\ t\in[0,1],\quad(6.30)$$

$$u^\epsilon(0,t)=u^\epsilon(\pi,t)=0\quad(6.31)$$

$$u^\epsilon(x,1)=\varphi_\epsilon(x)\quad(6.32)$$

To solve this problem we may apply the standard Euler's method to discrete it into the form

$$\frac{u^\epsilon(x,t_m)-u^\epsilon(x,t_{m+1})}{t_m-t_{m+1}}=-A_\epsilon(u^\epsilon(x,t_m))+B(\epsilon,t)(f(t_m,u^\epsilon(x,t_m))),\ t\in[0,1],\quad(6.33)$$

$$u^\epsilon(x,t_0)=u^\epsilon(x,1)=\varphi_\epsilon(x)\quad(6.34)$$

Here we use a uniform mesh $t_m=1-am(m=0,1,2,...)$ with the mesh size a. More clearly, we shall find $u^\epsilon(x,t_m)$ under the form

$$u^\epsilon(x,t_m)=\sum_{n=1}^{\infty}w_{n,m}\sin nx\quad(6.35)$$

where $w_{n,m}$ is computed by induction with m as follows

$$w_{n,0}=<\varphi_\epsilon(x),\sin nx>\quad,$$

$$w_{n,m+1}=(\epsilon+e^{-t_mn^2})^{\frac{t_{m+1}-t_m}{t_m}}\left(w_{n,m}-\frac{2}{\pi}\int\limits_{t_{m+1}}^{t_m}e^{(s-t_m)n^2}\left(\int\limits_0^\pi(u^\epsilon(x,t_m)+g(x,s))\sin nxdx\right)ds\right).$$

For simple computation, we shall find the regularized solution $u^\epsilon\left(x,\frac{999}{1000}\right)\equiv u_\epsilon(x)$ having the following form

$$u^\epsilon(x)=v_m(x)=w_{1,m}\sin x+w_{6,m}\sin 6x$$

where

$$v_1(x)=(\epsilon+1)e\sin x$$

$$w_{1,1}=(\epsilon+1)e,w_{6,1}=0,$$

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and

$$\begin{cases} a = \frac{1}{5000} \\ t_m = 1 - am \quad m = 1, 2, ..., 5 \\ w_{i,m+1} = \\ = (\epsilon + e^{-t_m i^2})^{\frac{t_{m+1}-t_m}{t_m}} \left(w_{i,m} - \frac{2}{\pi} \int\limits_{t_{m+1}}^{t_m} e^{(s-t_m)i^2} \left(\int\limits_0^\pi \left(v_m^4(x) + g(x,s) \right) \sin ixdx \right) ds \right), i = 1, 6. \end{cases}$$

Put $a_\epsilon = \|u_\epsilon - u\|$ the error between the regularization solution u_ϵ and the exact solution u. Letting $\epsilon = \epsilon_1 = 10^{-3}, \epsilon = \epsilon_2 = 10^{-7}, \epsilon = \epsilon_3 = 10^{-11}$, we have the first table

ϵ	u^ϵ	a_ϵ
10^{-3}	$2.718118645 \sin(x) - 0.005612885749 \sin(6x)$	0.002585244486
10^{-4}	$2.715807105 \sin(x) - 0.005488275207 \sin(6x)$	0.0002723211648
10^{-11}	$2.715552177 \sin(x) - 0.005518178192 \sin(6x)$	0.00004317829056

If we apply the method given in [63], we have the another approximation solution as follows

$$u_\epsilon(x, \frac{999}{1000}) = v_m(x) = w_{1,m} \sin x + w_{3,m} \sin 3x$$

where

$$v_1(x) = (\epsilon + 1)e \sin x$$

$$w_{1,1} = (\epsilon + 1)e, w_{3,1} = 0,$$

$$\begin{cases} a = \frac{1}{5000} \\ t_m = 1 - am \quad m = 1, 2, ..., 5 \\ w_{i,m+1} = \\ = e^{(t_m - t_{m+1})\frac{i^2}{1+\epsilon i^2}} w_{i,m} - \frac{2}{\pi} \int\limits_{t_{m+1}}^{t_m} e^{s - t_{m+1} - \frac{(t_m - t_{m+1})\epsilon i^4}{1+\epsilon i^2}} \left(\int\limits_0^\pi \left(v_m^4(x) + g(x,s) \right) \sin ixdx \right) ds, i = 1, 3. \end{cases}$$

We have the second error table

ϵ	u^ϵ	a_ϵ
10^{-3}	$2.718071080 \sin(x) - 0.005676570519 \sin(3x)$	0.006205188635
10^{-4}	$2.715756679 \sin(x) - 0.005544174983 \sin(3x)$	0.005547490740
10^{-11}	$2.715499520 \sin(x) - 0.005529451769 \sin(3x)$	0.005529838340

If we apply the method QBV given in [80], we have the approximation solution as follows

$$u_\epsilon(x, \frac{999}{1000}) = v_m(x) = w_{1,m} \sin x + w_{3,m} \sin 3x$$

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where

$$v_1(x) = (\epsilon + 1)e \sin x$$

$$w_{1,1} = (\epsilon + 1)e, w_{3,1} = 0,$$

$$\begin{cases} a = \frac{1}{5000} \\ t_m = 1 - am \quad m = 1, 2, ..., 5 \\ w_{i,m+1} = \frac{e^{-t_{m+1}i^2}}{\epsilon + e^{-t_m i^2}} w_{i,m} - \frac{2}{\pi} \int\limits_{t_{m+1}}^{t_m} \frac{e^{-t_{m+1}i^2}}{\epsilon^{s/t_m} + e^{-si^2}} \left(\int\limits_0^\pi \left(v_m^4(x) + g(x, s)\right) \sin ixdx \right) ds, \quad i = 1, 2, 3. \end{cases}$$

We have the third table

ϵ	u^ϵ	a_ϵ
10^{-3}	$2.677010541 \sin(x) - 0.00001660800099 \sin(3x)$	0.03855436757
10^{-4}	$2.706890214 \sin(x) - 0.00002685739642 \sin(3x)$	0.008674732576
10^{-11}	$2.710234914 \sin(x) - 0.00004360860072 \sin(3x)$	0.005330169394

Looking at three above tables in comparison with three other methods, we can see the error results of the first Table are smaller than theirs in the second Table and third table. This shows that our approach has a nice regularizing effect and give a better approximation in comparison with the previous method in, for example [63, 80].

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ABSTRACT

ON A MULTI-DIMENSIONAL SINGULAR STOCHASTIC CONTROL PROBLEM: THE PARABOLIC CASE

by

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December 2015

Advisor: Prof. Jose Luis Menaldi

Major: Mathematics (Applied)

Degree: Doctor of Philosophy

This dissertation considers a stochastic dynamic system which is governed by a multidimensional diffusion process with time dependent coefficients. The control acts additively on the state of the system. The objective is to minimize the expected cumulative cost associated with the position of the system and the amount of control exerted. It is proved that Hamilton-Jacobi-Bellman's equation of the problem has a solution, which corresponds to the optimal cost of the problem. We also investigate the smoothness of the free boundary arising from the problem.

In the second part of the dissertation, we study the backward parabolic problem for a nonlinear parabolic equation of the form $u_t + Au(t) = f(t, u(t)), u(T) = \varphi$, where A is a positive self-adjoint unbounded operator and f is a Lipschitz function. The problem is ill-posed, in the sense that if the solution does exist, it will not depend continuously on the data. To regularize the problem, we use the quasi-reversibility method to establish a modified problem. We present approximated solutions that depend on a small parameter $\epsilon > 0$ and give error estimates for our regularization. These results extend some work on the nonlinear backward problem. Some numerical examples are given to justify the theoretical analysis.

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